Bounded Derivations on Uniform Roe Algebras

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Abstract

In this paper we give conditions on a space X to give a positive answer to the question of whether or not all the derivations of the uniform Roe algebra on a space X are inner. Specifically, if a space X has a metric d under which (X, d) is a metric space with bounded geometry having property A, then all derivations are inner.

1 Introduction

Uniform Roe algebras are constructed from a metric space (or coarse space) X acting on $\ell^2(X)$. First we start with finite propagation operators in $\mathscr{B}(\ell^2(X))$. In this context "finite propagation" roughly means that, looking at an operator T as a $X \times X$ matrix, the nonzero entries of T are a finite distance from the diagonal. Thus, uniform Roe algebras reflect the coarse structure of a metric space and allow us to study the large scale geometry of the metric space while remaining "small" enough to have interesting K-theory.

The theory of derivations have a rich history in both mathematics and physics. The study of derivations on operator algebras arose in quantum mechanics. In the 1940's mathematicians asked if Heisenberg's commutation relations of momentum and position were given by bounded linear operators on a Banach space. Given a Banach space E letting $\mathscr{B}(E)$ be the algebra of bounded linear operators on E; for $A, B \in \mathscr{B}(E)$, $\delta_B(A) := \mathrm{ad}(B) = [B, A] = BA - AB$ defines a bounded linear operator on $\mathscr{B}(E)$ satisfying Leibniz rule. Thus, the theory of commutators became an important part of the theory of derivations.

Naturally the question of whether or not all bounded derivations are of this form, which we refer to as being "inner", is of concern. However, this is not the case, even if we restrict to C*-algebras (see [4] for examples). On the other hand, for the case of uniform Roe algebras, we will show that all bounded derivations are in fact inner; that is, if our space X is sufficiently "nice".

2 Preliminaries

For a Hilbert space $\mathcal{H}$, we denote the space of bounded operators on $\mathcal{H}$ by $\mathscr{B}(\mathcal{H})$. By $\ell^2(X)$, we mean the square summable complex sequences indexed by X. For a Banach space A, we denote by A_1 the norm closed unit ball of A. To denote the canonical basis of $\ell^2(X)$, we shall use $(\vartheta_x)_{x\in X}$ as we reserve δ for derivations. The support of an operator $T \in \mathscr{B}(\ell^2(X))$ is defined as

$$\operatorname{supp}(T) := \{(x, y) \in X \times X : \langle T\vartheta_x, \vartheta_y\rangle \neq 0\}.$$

Definition 2.1. For a metric space (X, d) we say that an operator T has propagation of at most R if $\langle T\vartheta_x, \vartheta_y\rangle = 0$ whenever $d(x, y) > R$ for all $(x, y) \in X \times X$ and write $\operatorname{prop}(T) \leq R$. The set of all operators of propagation at most R is denoted as

$$\mathbb{C}_u^R[X] := \{T \in \mathscr{B}(\ell^2(X)) : \operatorname{prop}(T) \leq R\}.$$

The *algebraic uniform Roe algebra* is defined as

$$\mathbb{C}_u[X] := \{T \in \mathscr{B}(\ell^2(X)) : \operatorname{prop}(T) < \infty\}$$

and the *uniform Roe algebra* is defined as the norm closure of the algebraic uniform Roe algebra; i.e.

$$C_u^*(X) := \overline{\{T \in \mathscr{B}(\ell^2(X)) : \operatorname{prop}(T) < \infty\}}^{\|\cdot\|}.$$

Definition 2.2 (bounded geometry)**.** Let (X, d) be a metric space. Then X is said to have *bounded geometry* if for every $R \geq 0$ there exists an $N_R \in \mathbb{N}$ such that for all $x \in X$, the ball of radius R about x has at most N_R elements.

Remark 2.3. Note that, for a metric space (X, d) with bounded geometry, $C_b(X) = \ell^\infty(X)$ which we will use interchangeably. Additionally, we may view an element $a \in \ell^\infty(X)$ as an element $T^{(a)} \in C_u^*(X)$ by setting

$$T_{xy}^{(a)} := \begin{cases} a(x) & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

Moreover, $\|a\|_{\ell^\infty} = \sup_{x\in X} |a(x)| = \sup_{\|\xi\|=1} \left\|T^{(a)}\xi\right\|_{\ell^2} = \left\|T^{(a)}\right\|$. To keep notation simple we will denote $T^{(a)}$ by a.

Lemma 2.4. *Let (X, d) be a metric space with bounded geometry. Then there* *are at most* $2N_R - 1$ *(where* $N_R \in \mathbb{N}$ *is from the previous definition) partially defined bijections, say* t_n *where* $n \in \{1, \ldots, 2N_R - 1\}$*, such that for any* $x \in X$ *if* $d(x, y) \leq R$ *then there exists a unique* $n \in \{1, \ldots, 2N_R - 1\}$ *such that* $t_n(x) = y$*.*

Proof. Define $E_0 = \emptyset$ and $E_R = \{(x, y) \in X \times X : d(x, y) \leq R\}$. Then for $n \in \{1, \ldots, 2N_R - 1\}$ define E_n to be a maximal subset of $E_R \setminus \cup_{i=0}^{n-1} E_i$ such that the coordinate projections π_1, π_2 are injective on E_n. Let $\pi_1(E_n) = A_n$ and $\pi_2(E_n) = B_n$ Since π_1 and π_2 are injective on E_n we may define bijections

$$\pi_{1,n} : E_n \to \pi_1(E_n) = A_n \quad \text{and} \quad \pi_{2,n} : E_n \to \pi_2(E_n) = B_n.$$

Thus, we may define a bijection

$$t_n : A_n \xrightarrow{\pi_{1,n}^{-1}} E_n \xrightarrow{\pi_{2,n}} B_n.$$

Next, suppose for contradiction that $E_R \setminus \cup_{i=0}^{2N_R - 1} E_i \neq \emptyset$. Then there exists a (x_0, y_0), where $d(x_0, y_0) \leq R$ such that for all $n \in \{1, \ldots, 2N_R - 1\}$, $(x_0, y_0) \notin E_n$. Note that, given $n \in \{1, \ldots, 2N_R - 1\}$ if $(x_0, y_0) \notin E_n$ then there exists a

$$(x_0, y_n) \in E_n,\ y_n \neq y_0 \quad \text{or a} \quad (x_n, y_0) \in E_n,\ x_n \neq x_0$$

since E_n is the maximal subset of $E_R \setminus \cup_{i=0}^{n-1} E_i$ such that π_1, π_2 are injective. Define

$$A := \{y_i \in \overline{B_R(x_0)} : (x_0, y_i) \in E_i, y_i \neq y_0\}, \text{ and}$$

$$B := \{x_i \in \overline{B_R(y_0)} : (x_i, y_0) \in E_i, x_i \neq x_0\}$$

Since the E_n's are disjoint by construction, $y_i \neq y_j$ and $x_i \neq x_j$ for all $i \neq j$; $i, j \in \{1, \ldots, 2N_R - 1\}$. Note that $A \subseteq \overline{B_R(x_0)} \setminus \{y_0\}$ and $B \subseteq \overline{B_R(y_0)} \setminus \{x_0\}$. Hence, since X has bounded geometry, $|A|, |B| \leq N_R - 1$. However, this contradicts the maximality of the E_n's since

$$(x_0, y_n) \in E_n,\ y_n \neq y_0 \quad \text{or a} \quad (x_n, y_0) \in E_n,\ x_n \neq x_0$$

can occur at most $2N_R - 2$ times.

Thus, given (x, y) such that $d(x, y) \leq R$ there exists an $n \in \{1, \ldots, 2N_R - 1\}$ such that $(x, y) \in E_n$. Moreover, this n is unique since the E_n's are disjoint. Furthermore, for this n, $t_n(x) = y$. □

Lemma 2.5. *Let (X,d) be a metric space with bounded geometry. Then for $T \in \mathbb{C}_u^R[X]$ we have*

$$\|T\| \leq \sup_{x,y\in X} |T_{xy}|\,(2N_R - 1)$$

Proof. Let A_n and t_n be defined as in the previous lemma. Define a partial isometry, $v_n : \ell^2(X) \to \ell^2(X)$ by

$$v_n : \vartheta_x \mapsto \begin{cases} \vartheta_{t_n(x)} & \text{if } x \in A_n \\ 0 & \text{otherwise} \end{cases}$$

For $T \in \mathbb{C}_u^R[X]$ define $f_n(x) = T_{x\ t_n(x)}$ Then we may write $T = \sum_{n=1}^{2N_R-1} f_n v_n$. Thus,

$$\|T\| = \left\| \sum_{n=1}^{2N_R-1} f_n v_n \right\| \leq \sum_{n=1}^{2N_R-1} \|f_n\|\,\|v_n\| \leq \sup_{x,y\in X} |T_{xy}|\,(2N_R - 1)$$

□

Definition 2.6 (derivation)**.** By a derivation on a C^*-algebra $\mathscr{A}$ we shall mean a linear map $\delta : \mathscr{A} \to \mathscr{A}$ that satisfies Leibniz's rule. That is $\delta(ab) = \delta(a)b + b\delta(a)$.

Definition 2.7 (spatial)**.** A derivation δ of a C^*-algebra $\mathscr{A}$ acting on a Hilbert space $\mathcal{H}$ is *spatial* if there is a bounded operator $h \in \mathscr{B}(\mathcal{H})$ such that $\delta(a) = ha - ah = [h,a] = \text{ad } h(a)$.

Definition 2.8 (Concrete C*-algebra)**.** A *concrete C*-algebra* is a $*$-subalgebra of $\mathscr{B}(\mathcal{H})$ that is closed in the norm topology.

Note that by Kadison [5] Theorem 4, every bounded derivation on a concrete C^*-algebra is spatial. However, in our circumstance we are able to give a more direct (but less general) proof by slightly modifying a proof of Kaplansky.

Theorem 2.9 ([**?**])**.** *Every bounded derivation $\delta : C_u^*(X) \to C_u^*(X)$ is given by $[b,\cdot]$ for some $b \in \mathscr{B}(\ell^2(X))$.*

Proof. Let δ be an arbitrary derivation on $C_u^*(X)$, and let $p \in \ell^\infty(X) \subseteq C_u^*(X)$ be the rank one projection onto the n'th coordinate of X. Define:

$$y := [p, \delta(p)], \text{ and } \delta'(a) := \delta(a) - [a,y] \tag{1}$$

Then, $\delta(a) = \delta'(a) - [y, a]$. Note that:

$$\delta(p) = \delta(p^2) = p\delta(p) + \delta(p)p \implies p\delta(p)p = 0, \tag{2}$$

so that,

$$[p, y] = p\Big(p\delta(p) - \delta(p)p\Big) - \Big(p\delta(p) - \delta(p)p\Big)p = \delta(p). \tag{3}$$

Thus, $\delta'(p) = 0$.

Next, since $\delta'(ap) = \delta'(a)p$ for all $a \in C_u^*(X)$, the linear map $ap \mapsto \delta'(a)p$ is well defined on the left ideal $C_u^*(X)p$. Note that for any $a \in C_u^*(X)$, ap is equivalent to an operator

$$T_{xy}^{(ap)} := \begin{cases} a_{xy} & \text{if } y = n \\ 0 & \text{otherwise} \end{cases}$$

(recalling that p is the projection onto the n'th coordinate of X). Hence, $C_u^*(X)\,p \cong \ell^2(X)$, and so the restriction of δ' to $C_u^*(X)\,p$, $\delta' \restriction_{C_u^*(X)p}$, can be identified with a bounded operator in $\mathscr{B}(\ell^2(X))$. Thus, there exists an $h \in \mathscr{B}(\ell^2(X))$ such that $hap = \delta'(a)p$. Hence,

$$haxp = \delta'(ax)p = \delta'(a)xp + a\delta(x)p = \delta'(a)xp + ahxp \qquad \text{for all } x \in C_u^*(X)$$

$$\implies haxp - ahxp = \delta'(a)xp \implies [h, a]xp = \delta'(a)xp$$

$$\implies [h, a] = \delta'(a)$$

Therefore, $\delta(a) = \delta'(a) - [y, a] = [h, a] - [y, a] = [h - y, a]$ as was to be shown. □

3 A Containment Condition

In this section we define a condition to determine when an element $b \in \mathscr{B}(\mathcal{H})$ is in $C_u^*(X)$ by following [9] and [11] where $\mathcal{H} = \ell^2(X)$ and X is a metric space with bounded geometry having property A. Namely, for any $b \in \mathscr{B}(\mathcal{H})$

$$b \in C_u^*(X) \text{ if and only if } [b, f] = 0 \text{ for all } f \in VL_\infty(X)_1 \tag{4}$$

where $VL_\infty(X)$ will be defined in section 3.2. However, we will need some definitions and several lemmas to prove this statement.

3.1 Block Cutdowns

Lemma 3.1. *Let* $(\phi_j)_{j\in J}$ *be a family of positive contractions in* $C_b(X)$ *such that* $\left\|\sum_{j\in J}\phi_j^2\right\| < 1$*. Then for all* $b \in \mathscr{B}(\ell^2(X))$ *the sum* $\sum_{j\in J}\phi_j b\phi_j$ *converges strongly to an operator in* $\mathscr{B}(\ell^2(X))$*.*

Proof. First we consider the case when $b \geq 0$. Note that each ϕ_j is positive and so each $\phi_j b\phi_j \geq 0$. Thus, since finite sums of positive elements are again positive, $\sum_{j\in F}\phi_j b\phi_j \geq 0$ for all finite $F \subseteq X$. Moreover, for any finite sets $F_1, F_2 \subseteq X$ such that $F_1 \subseteq F_2$ we have

$$\sum_{j\in F_2}\phi_j b\phi_j - \sum_{j\in F_1}\phi_j b\phi_j = \sum_{j\in F_2\setminus F_1}\phi_j b\phi_j \geq 0$$

so

$$0 \leq \sum_{j\in F_1}\phi_j b\phi_j \leq \sum_{j\in F_2}\phi_j b\phi_j \text{ whenever } F_1 \subseteq F_2.$$

Next, since $\left\|\sum_{j\in F}\phi_j^2\right\| \leq 1$ for all finite F, for $\xi \in \ell^2(X)$ we have

$$\left\|\sum_{j\in F}\phi_j b\phi_j\xi\right\| \leq \left\|\sum_{j\in F}\phi_j \|b\| \phi_j\xi\right\| \leq \|b\| \left\|\sum_{j\in F}\phi_j^2\xi\right\| \leq \|b\| \|\xi\|.$$

Thus, since the net of partial sums is increasing and bounded above, $\sum_{j\in J}\phi_j b\phi_j$ converges in the strong operator topology to an operator in $\mathscr{B}(\ell^2(X))$.

For a general $b \in \mathscr{B}(\ell^2(X))$, by separating b into its real and imaginary parts $b = a + ic$, then separating its real and imaginary parts into their positive and negative parts $a = a^+ - a^-$ and $c = c^+ - c^-$ we may write

$$\sum_{j\in J}\phi_j b\phi_j = \sum_{j\in J}\phi_j a^+\phi_j - \sum_{j\in J}\phi_j a^-\phi_j + i\sum_{j\in J}\phi_j c^+\phi_j - i\sum_{j\in J}\phi_j c^-\phi_j.$$

By the positive case each of the sums on the right hand side strongly converges to an operator in $\mathscr{B}(\ell^2(X))$ and so $\sum_{j\in J}\phi_j b\phi_j$ converges strongly. □

Definition 3.2 ([9], Definition 2.2)**.** Let (X, d) be a proper metric space and let $\mathcal{H} = \ell^2(X)$. Given a family $(e_j)_{j\in J}$ of positive contractions in $C_b(X)$ with pairwise disjoint supports, define the *block cutdown* map

$$\theta_{(e_j)_{j\in J}} : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H}) \text{ by } \theta_{(e_j)_{j\in J}}(a) := \sum_{j\in J} e_j a e_j$$

Definition 3.3 (strong* topology)**.** A net $\{a_\alpha\}$ converges in the *strong* topology* to $a \in \mathscr{B}(\mathcal{H})$ if and only if both

$$a_\alpha \xrightarrow{SOT} a \qquad \text{and} \qquad a_\alpha^* \xrightarrow{SOT} a^*$$

where SOT stands for the strong operator topology.

Definition 3.4 (L-Lipschitz)**.** A function $f \in C_b(X)$ is *L-Lipschitz* if for all $x, y \in X$ we have $|f(x) - f(y)| \leq L \cdot d(x, y)$

Before we continue let us fix the following notation:
For $L, \epsilon > 0$, define

$$\text{Commut}(L, \epsilon) := \{b \in \mathscr{B}(\ell^2(X)) : \|[b, g]\| < \epsilon \text{ for all } L\text{−Lipschitz } g \in C_b(X)_1\}.$$

For a set $G \subset \mathscr{B}(\mathcal{H})$ we denote the commutant of G by G'; that is,

$$G' := \{T \in \mathscr{B}(\mathcal{H}) : TS = ST \text{ for all } S \in G\}.$$

Lastly, for a metric space X we say that a family $(Y_j)_{j\in J} \subseteq X$ is R-disjoint if $d(Y_j, Y_i) > R$ whenever $j \neq i$.

Definition 3.5 (Conditional Expectation)**.** Let B be a C^*-subalgebra of the C^*-algebra A. A *conditional expectation*, E is a completely positive contractive projection

$$E : A \to B \text{ such that } E(b_1 a b_2) = b_1 E(a) b_2 \text{ for all } b_1, b_2 \in B \text{ and } a \in A.$$

Lemma 3.6 ([9], Lemma 4.1)**.** *Let G be a $*$-closed, strong* compact, commutative subgroup of the unitary operators. Then there exists a unique conditional expectation*
$E_G : \mathscr{B}(\mathcal{H}) \to G'$ *such that:*

i) The restriction of E_G to the unit ball of $\mathscr{B}(\mathcal{H})$ is weak operator topology continuous, and

ii) $\|E_G(a) - a\| \leq \sup_{u \in G} \|[a, u]\|$ for all $a \in \mathscr{B}(\mathcal{H})$

Proof. Let $a \in \mathscr{B}(\mathcal{H})$ be fixed but arbitrary and define a map $\varphi_a : G \to \mathscr{B}(\mathcal{H})$ by $u \mapsto u^* a u$. Endowing G with the strong* topology and $\mathscr{B}(\mathcal{H})$ with the weak operator topology this map is continuous. Indeed, given $\epsilon > 0$ let $\{u_n\}$ be a net in G converging in the strong* topology to $u \in G$, and let $\phi \in \mathcal{H}^*,\ \xi \in \mathcal{H}$ be fixed but arbitrary. Since $u_n \xrightarrow{\text{sot}} u$ there exists $n_1,\ n_2$ such that

$$\|(u_n - u)\xi\|_{\ell^2} < \frac{\epsilon}{2\,\|\phi\|\,\|a\|} \qquad \text{and} \qquad \|(u_n - u)\eta\|_{\ell^2} < \frac{\epsilon}{2\,\|\xi\|\,\|a\|}$$

whenever $n \geq n_1$ and $n \geq n_2$ where $\|\eta\| = \|\phi\|$ is given by the Riesz Representation Theorem. Thus,

$$|\phi(u_n^* a u_n \xi - u^* a u \xi)| \leq |\phi\,(u_n^* a)(u_n\xi - u\xi))| + |\phi\,((u_n^* - u^*)(au\xi))|$$

$$\leq \|\phi\|\,\|(u_n^* a)(u_n\xi - u\xi)\| + |\langle (u_n^* - u^*)(au\xi), \eta\rangle|$$

$$\leq \|\phi\|\,\|u_n^*\|\,\|a\|\,\|(u_n - u)\xi\| + |\langle au\xi, (u_n - u)\eta\rangle|$$

$$\leq \|\phi\|\,\|u_n^*\|\,\|a\|\,\|(u_n - u)\xi\| + \|a\|\,\|u\|\,\|\xi\|\,\|(u_n - u)\eta\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

where the last line is given by the Cauchy-Schwartz inequality. Note that the above also shows that the family $\{\varphi_a\}_{\|a\|\leq 1}$ is weakly equicontinuous. Let μ_G be the unique normalized Haar measure for G with the strong* topology. Since our map φ_a is continuous we may define $E_G(a)$ to be the Pettis integral of φ_a; that is,

$$E_G(a) := \text{WOT-}\int_G u^* a u \,\mathrm{d}\,\mu_G(u).$$

Now we check that E_G satisfies the conditions of being a conditional expectation. Recall that a Riemann sum for this integral has the form $S_P = \sum_{i=1}^n u_i^* a u_i \mu_G(G_i)$ where the partition P is given by a finite collection $\{G_1, \ldots, G_n\}$ of disjoint subsets of G such that $G = \sqcup_{i=1}^n G_i$ and $u_i \in G_i$. Observe that for any Riemann sum and any $f \in G$,

$$\left(\sum_{i=1}^n u_i^* a u_i \mu_G(G_i)\right) f = \sum_{i=1}^n u_i^* a u_i f \mu_G(G_i)$$

$$= f \sum_{i=1}^n (u_i f)^* a (u_i f) \mu_G(G_i) = f \sum_{i=1}^n (u_i f)^* a (u_i f) \mu_G(G_i f)$$

by the invariance of Haar measure. Since both of these Riemann sums converge to $\text{WOT-}\int_G u^* a u \,\mathrm{d}\,\mu_G(u)$, $E_G(a)$ commutes with every element of G for all $a \in \mathscr{B}(\mathcal{H})$. We now use this to show that E_G is a projection. Observe that,

$$E_G(E_G(a)) = \text{WOT-}\int_G u^* E_G(a) u \,\mathrm{d}\,\mu_G(u)$$

$$= \text{WOT-}\int_G u^* u E_G(a)\,\mathrm{d}\,\mu_G(u) = E_G(a)\mu_G(G) = E_G(a).$$

Hence, by [3], Theorem 1.5.10 we need only show that E_G is contractive. Observe that,

$$\|E_G\| = \sup_{\|a\|\le 1} \|E_G(a)\| = \sup_{\|a\|\le 1} \left\| \text{WOT-}\int_G u^* a u\,\mathrm{d}\,\mu_G(u) \right\|$$

$$\le \sup_{\|a\|\le 1} \text{WOT-}\int_G \|u^*\|\,\|a\|\,\|u\|\,\mathrm{d}\,\mu_G(u) \le \mu_G(G) = 1$$

Now we show that the restriction of E_G to the unit ball $(\mathscr{B}(\mathcal{H}))_1$ is weak operator topology continuous. Recall that the family $\{\varphi_a\}_{\|a\|\le 1}$ is weakly equicontinuous. Thus, given $\epsilon > 0$ and $\xi, \eta \in \ell^2(X)$ there exists a $\delta_1, \delta_2 > 0$ such that

$$\left|\left\langle (u_i^* a u_i - u_j^* a u_j)\xi, \eta \right\rangle\right| < \epsilon \text{ whenever}$$

$$u_i, u_j \in \{u \in G : \|(u - v)\xi\| < \delta_1\} \cap \{u \in G : \|(u - v)\eta\| < \delta_2\}$$

for any $\|a\| \le 1$. Hence, if

$$P = \{G_i\}_{i=1}^n, \bigsqcup_{i=1}^n G_i = G \text{ and } Q = \{G_j\}_{j=1}^m, \bigsqcup_{j=1}^m G_j = G$$

are any two (finite) partitions of G such that each $G_i \in P, G_j \in Q$ is $*$-closed, and

$$u_k, u_l \in G_i, \quad u_k, u_l \in \{u \in G : \|(u - v)\xi\| < \delta_1\} \cap \{u \in G : \|(u - v)\eta\| < \delta_2\}$$

$$u_k, u_l \in G_j, \quad u_k, u_l \in \{u \in G : \|(u - v)\xi\| < \delta_1\} \cap \{u \in G : \|(u - v)\eta\| < \delta_2\}$$

(such partitions are possible since G is strong* compact) then on their refinement $K = \{G_k\}_{k=1}^{\ell}$ we have

$$\left| \sum_{k=1}^{\ell} \left\langle \left(u_{k_1}^* a u_{k_1} - u_{k_2}^* a u_{k_2} \right) \xi, \eta \right\rangle \mu_G(G_k) \right|$$

$$\le \sum_{k=1}^{\ell} \epsilon \mu_G(G_k) = \epsilon.$$

Thus, the integral defining E_G when restricted to the unit ball of $\mathscr{B}(\mathcal{H})$ can be uniformly approximated in the weak operator topology by finite Riemann sums. Moreover, for any partition P, the map $a \mapsto \sum_{i=1}^{n} u_i^* a u_i \mu_G(G_i)$ is weakly continuous. Hence, E_G is weakly continuous on the unit ball of $\mathscr{B}(\mathcal{H})$.

We now show ii). First note that

$$\|[a,u]\| = \|u(u^*au - a)\| \leq \|u^*au - a\| \text{ and } \|u^*au - a\| = \|u^*(au - ua)\| \leq \|[a,u]\|$$

for all $u \in G$ so that $\|[a,u]\| = \|u^*au - a\|$. Then, for any $\xi, \eta \in \mathcal{H}$ we have

$$|\langle (E_G(a) - a)\,\xi, \eta\rangle| = |\langle E_G(a)\xi, \eta\rangle - \langle a\xi, \eta\rangle|$$

$$= \left| \int_G \langle u^*au\xi, \eta\rangle \,\mathrm{d}\,\mu_G(u) - \int_G \langle a\xi, \eta\rangle \,\mathrm{d}\,\mu_G(u) \right| = \left| \int_G \langle (u^*au - a)\,\xi, \eta\rangle \,\mathrm{d}\,\mu_G(u) \right|$$

$$\leq \int_G |\langle (u^*au - a)\,\xi, \eta\rangle| \,\mathrm{d}\,\mu_G(u) \leq \int_G \|u^*au - a\| \,\|\xi\|\, \|\eta\| \,\mathrm{d}\,\mu_G(u)$$

$$= \int_G \|[a,u]\| \,\|\xi\|\, \|\eta\| \,\mathrm{d}\,\mu_G(u) \leq \sup_{u \in G} \|[a,u]\| \,\|\xi\|\, \|\eta\|\,.$$

Let $\xi \in \mathcal{H}$ and take $\eta_\xi = \frac{(E_G(a)-a)\xi}{\|(E_G(a)-a)\xi\|}$ whenever $(E_G(a)-a)\xi \neq 0$, then we have

$$\|(E_G(a) - a)\xi\| = |\langle (E_G(a) - a)\xi, \eta_\xi\rangle| \leq \sup_{u \in G} \|[a,u]\| \,\|\xi\|$$

for all $\xi \in \mathcal{H}$. Thus, $\|E_G(a) - a\| \leq \sup_{u \in G} \|[a,u]\|$ for all $a \in \mathscr{B}(\mathcal{H})$ since a was arbitrary.

Lastly, we show uniqueness. Suppose that $E : \mathscr{B}(\mathcal{H}) \to G'$ is another conditional expectation that is weakly continuous on $(\mathscr{B}(\mathcal{H}))_1$. Then, for $a \in (\mathscr{B}(\mathcal{H}))_1$ we have

$$\begin{aligned}
E_G(a) &= E(E_G(a)) && (E \text{ fixes } G') \\
&= E\left(\text{WOT-}\textstyle\int_G u^*au \,\mathrm{d}\,\mu_G(u)\right) \\
&= \text{WOT-}\textstyle\int_G E(u^*au) \,\mathrm{d}\,\mu_G(u) && (\text{WOT-continuity of } E \restriction_{\mathscr{B}(\mathcal{H})_1}) \\
&= \text{WOT-}\textstyle\int_G u^*E(a)u \,\mathrm{d}\,\mu_G(u) && (G \subseteq G' \text{ and } E \text{ is a conditional expectation}) \\
&= E_G(E(a)) \\
&= E(a) && (E_G \text{ fixes } G')
\end{aligned}$$

Thus, $E = E_G$.

□

Corollary 3.7 ([9], Corollary 4.2)**.** *Let $D \subset \mathscr{B}(\mathcal{H})$ be an atomic abelian von Neumann algebra. Then there is a unique conditional expectation $E_D : \mathscr{B}(\mathcal{H}) \to D'$ whose restriction to the unit ball is weakly continuous and satisfies*

$\|E_D(a) - a\| \leq \sup_{x \in D, \|x\| \leq 1} \|[a, x]\|$ *for all* $a \in \mathscr{B}(\mathcal{H})$.

Proof. Let $\mathbb{1}_D$ be the identity of D. Then, D is generated by a family of orthogonal projections $(p_j)_{j \in J}$ whose sum converges strongly to $\mathbb{1}_D$. Define

$$G := \left\{ \sum_{j \in J} (-1)^{\alpha_j} p_j : (\alpha_j)_{j \in J} \in (\mathbb{Z}/2)^J \right\}.$$

Note that G is strong* compact as it is homeomorphic to $(\mathbb{Z}/2)^J$ with the product topology via $\varphi(u_\alpha) = \alpha$ where $\alpha = (\alpha_j)_{j \in J}$ and $u_\alpha = \sum_{j \in J} (-1)^{\alpha_j} p_j$. Hence, it satisfies the conditions of Lemma 3.6. Moreover, G generates D so that $G' = D'$. Thus, $E_G : \mathscr{B}(\ell^2(X)) \to D'$ is the conditional expectation with the properties of 3.6. □

Corollary 3.8 ([9], Corollary 4.3)**.** *Let:*

1. (X, d) *be a metric space with bounded geometry,*
2. $a \in \mathrm{Commut}(L, \epsilon) \subset \mathscr{B}(\mathcal{H})$ *for some* $L, \epsilon > 0$ *where* $\mathcal{H} = \ell^2(X)$, *and*
3. $(e_j)_{j \in J}$ *be a family of positive contractions from* $C_b(X)$ *with* $(2L^{-1})$*-disjoint supports.*

Define $e := \sum_{j \in J} e_j$. *Then,*

$$\left\| eae - \theta_{(e_j)_{j \in J}}(a) \right\| \leq \epsilon$$

Proof. Set $A_j = \mathrm{supp}(e_j)$ Then we may find a family of pairwise orthogonal projections $(p_j)_{j \in J} \subset C_b(X)$ where each p_j is supported on A_j and acts as the identity on e_j. Define D to be the von Neumann subalgebra generated by this family of projections where $\mathbb{1}_D$ is the strong limit of $\sum_{j \in J} p_j$. Let $E_D : \mathscr{B}(\mathcal{H}) \to D'$ be the unique conditional expectation given by Corollary 3.7.

Note that since G is homeomorphic to $(\mathbb{Z}/2)^J$, if μ_Γ is the unique normalized Haar measure on $(\mathbb{Z}/2)^J = \Gamma$, then

$$\mu_G(H) = \mu_\Gamma\big(\{\alpha \in (\mathbb{Z}/2)^J : u_\alpha \in H\}\big) \quad \text{for all} \quad H \subseteq G.$$

Next, for each pair $(i, j) \in J \times J$ let

$$\Gamma^+_{(i,j)} = \{\alpha \in \Gamma : \alpha_i = \alpha_j\} \ \text{ and } \ \Gamma^-_{(i,j)} = \{\alpha \in \Gamma : \alpha_i \neq \alpha_j\}$$

Observe that, $\Gamma^+_{(i,j)} \sqcup \Gamma^-_{(i,j)} = \Gamma$. Moreover,

$$\mu_\Gamma(\Gamma^+_{(i,j)}) = 1/2 = \mu_\Gamma(\Gamma^-_{(i,j)}) \quad \text{whenever} \quad i \neq j,\ \Gamma^+_{(j,j)} = \Gamma, \quad \text{and} \quad \Gamma^-_{(j,j)} = \emptyset$$

Hence, for $b \in \mathscr{B}(\mathcal{H})$

$$E_D(b) = \text{WOT-}\int_G u^* b u \,\mathrm{d}\,\mu_G(u)$$

$$= \text{WOT-}\int_G \left(\sum_{i\in J}(-1)^{\alpha_i} p_i\right) b \left(\sum_{j\in J}(-1)^{\alpha_j} p_j\right) \mathrm{d}\,\mu_G(u_\alpha)$$

$$= \text{WOT-}\int_G \sum_{i,j\in J} (-1)^{\alpha_i+\alpha_j} p_i b p_j \,\mathrm{d}\,\mu_G(u_\alpha)$$

$$= \sum_{i,j\in J} p_i b p_j \int_\Gamma (-1)^{\alpha_i+\alpha_j} \,\mathrm{d}\,\mu_\Gamma(\alpha)$$

$$= \sum_{j\in J} p_j b p_j \int_{\Gamma^+_{(j,j)}} 1 \,\mathrm{d}\,\mu_\Gamma(\alpha) + \sum_{\substack{i,j\in J\\ i\neq j}} p_i b p_j \left(\int_{\Gamma^+_{(i,j)}} 1 \,\mathrm{d}\,\mu_\Gamma(\alpha) - \int_{\Gamma^-_{(i,j)}} 1 \,\mathrm{d}\,\mu_\Gamma(\alpha)\right)$$

$$= \sum_{j\in J} p_j b p_j$$

Then, since e_j and p_j are both supported on A_j and these supports are disjoint, we have

$$p_j \left(\sum_{i\in J} e_i\right) = e_j = \left(\sum_{i\in J} e_i\right) p_j$$

Thus,

$$E_D(eae) = \sum_{j\in J} p_j \left(\sum_{i\in J} e_i\right) a \left(\sum_{i\in J} e_i\right) p_j = \sum_{j\in J} e_j a e_j = \theta_{(e_j)_{j\in J}}(a) \qquad (5)$$

Next, note that for any $f \in D_1$, f is constant on each A_j since D is generated by the p_j's. Moreover, since the A_j's are $(2L^{-1})$-disjoint we may find an $\tilde{f} \in C_b(X)_1$ that is L-Lipschitz and such that $\tilde{f}$ agrees with f on the A_j's; that is, $\mathbb{1}_D \tilde{f} = f = \tilde{f}\mathbb{1}_D$. Indeed, if $x \in A_j$ and $y \in A_i$, $j \neq i$ then $d(x,y) > 2L^{-1}$ and so

$$\left|\tilde{f}(x) - \tilde{f}(y)\right| \leq \left|\tilde{f}(x)\right| + \left|\tilde{f}(y)\right| \leq 2 = L(2L^{-1}) \leq L\ d(x,y).$$

Additionally, if $\tilde{f}$ linearly interpolates between the A_j's then $\tilde{f}$ is L-Lipschitz. Notice that, since each $e_j \in C_b(X)_1$ (since the e_j's are contractive) and they have disjoint supports, $e = \sum_{j\in J} e_j \in C_b(X)_1$. Thus, $\tilde{f}$ commutes with e since both are in $C_b(X)$. Observe that,

$$\|[f, eae]\| = \|f(eae) - (eae)f\| = \left\|\tilde{f}eae - eae\tilde{f}\right\| = \left\|e\tilde{f}ae - ea\tilde{f}e\right\|$$

$$= \|e\| \left\|\tilde{f}a - a\tilde{f}\right\| \|e\| < \epsilon$$

since $\tilde{f} \in C_b(X)_1$ is L-Lipschitz and $a \in \mathrm{Commut}(L, \epsilon)$. Then since $f \in D_1$ was arbitrary, $\sup\limits_{f\in D_1} \|[f, eae]\| \leq \epsilon$. Therefore,

$$\left\|eae - \theta_{(e_j)_{j\in J}}(a)\right\| \overset{(5)}{=} \|E_D(eae) - eae\| \overset{3.6}{\leq} \sup_{f\in D_1} \|[f, eae]\| \leq \epsilon$$

as was to be shown. ☐

3.2 Property A and Proof of (4)

Definition 3.9 ([9], Definition 2.6)**.** A bounded sequence $(f_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, C_b(X))$ is very Lipschitz if for each $L > 0$ there is an n_L such that f_n is L-Lipschitz whenever $n \geq n_L$. Let $VL(X)$ be the set of all very Lipschitz bounded sequences. Define

$$VL_\infty(X) := {}^{VL(X)}\!\Big/\!_{\{(f_n)_{n\in\mathbb{N}} \in VL(X)\,:\, \lim\limits_{n\to\infty} \|f_n\|_{C_b(X)} = 0\}},$$

and for $\mathcal{H} = \ell^2(X)$

$$(\mathscr{B}(\mathcal{H}))_\infty := {}^{\ell^\infty(\mathbb{N}, \mathscr{B}(\mathcal{H}))}\!\Big/\!_{\{(f_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, \mathscr{B}(\mathcal{H}))\,:\, \lim\limits_{n\to\infty} \|f_n\|_{op} = 0\}}.$$

Lemma 3.10. *$VL(X)$ is a C^*-subalgebra of $\ell^\infty(\mathbb{N}, C_b(X))$.*

Proof. That $VL(X)$ is closed under addition, scaler multiplication, and involution is clear. To show that $VL(X)$ is closed under multiplication let $f, g \in VL(X)$ and $L > 0$ be given. Take n large enough so that

$$|f_n(x) - f_n(y)| \leq \frac{L}{2\|g\|} \cdot d(x, y) \quad \text{and} \quad |g_n(x) - g_n(y)| \leq \frac{L}{2\|f\|} \cdot d(x, y).$$

Observe that

$$|f_n(x)g_n(x) - f_n(y)g_n(y)| = |f_n(x)g_n(x) - f_n(y)g_n(x) - f_n(y)g_n(x) + f_n(y)g_n(y)|$$

$$\leq |f_n(x) - f_n(y)| \, \|g\| + \|f\| \, |g_n(x) - g_n(y)| \leq L \cdot d(x,y)$$

Next, let $\{f^\alpha\}_{\alpha \in A}$ be a net in $VL(X)$ converging to $f \in \ell^\infty(\mathbb{N}, C_b(X))$. Since $f^\alpha \to f$, given $\epsilon > 0$ there exists an α_0 such that

$$\sup_{n \in \mathbb{N}} \sup_{x \in X} |f_n^\alpha(x) - f_n(x)| = \|f^\alpha - f\|_{\ell^\infty(\mathbb{N}, C_b(X))} < \epsilon \quad \text{whenever} \quad \alpha \geq \alpha_0.$$

Fix $\alpha > \alpha_0$. For this α there exists an n_L such that $|f_n^\alpha(x) - f_n^\alpha(y)| \leq L \cdot d(x,y)$ whenever $n \geq n_L$. Observe that

$$|f_n(x) - f_n(y)| = |f_n(x) - f_n^\alpha(x) + f_n^\alpha(x) - f_n^\alpha(y) + f_n^\alpha(y) - f_n(y)|$$

$$\leq |f_n(x) - f_n^\alpha(x)| + |f_n^\alpha(x) - f_n^\alpha(y)| + |f_n^\alpha(y) - f_n(y)| < 2\epsilon + L \cdot d(x,y)$$

whenever $n \geq n_L$, so that $f \in VL(X)$. □

Note that $VL_\infty(X) \subseteq (\mathscr{B}(\mathcal{H}))_\infty$, and both are C^*-algebras under the quotient norm

$$\|f\|_{ql} = \limsup_{n \to \infty} \|f_n\|_{op} \text{ where } f = (f_n)_{n \in \mathbb{N}}.$$

Lemma 3.11 ([11],Lemma 3.5)**.** *Let $b \in \mathscr{B}(\ell^2(x))$ and $\epsilon > 0$ be given. Then $\|[b,f]\|_{ql} < \epsilon$ for every $f \in VL_\infty(X)_1$ if and only if there exists some $L > 0$ such that $b \in \mathrm{Commut}(L, \epsilon)$.*

Proof. For the reverse direction let L be fixed but arbitrary. Observe that for any $f = (f_n)_{n \in \mathbb{N}} \in VL_\infty(X)_1$, we may assume that $f_n \in C_b(X)_1$ for all $n \in \mathbb{N}$. Moreover, there exists an n_L such that f_n is L-Lipschitz for all $n \geq n_L$. Thus, since $b \in \mathrm{Commut}(L, \epsilon)$, $\|[b, f_n]\| < \epsilon$ for all $n \geq n_L$ and so $\|[b,f]\|_{ql} < \epsilon$.
For the forward direction suppose for contradiction that $\|[b,f]\|_{ql} < \epsilon$ for all $f \in VL_\infty(X)$ but, for all $L > 0$, $b \notin \mathrm{Commut}(L, \epsilon)$. Then, for each n there exists a $\frac{1}{n}$-Lipschitz function $f_n \in C_b(X)_1$ such that $\|[b, f_n]\| \geq \epsilon$ for all n. However, $f = (f_n)_{n \in \mathbb{N}} \in VL_\infty$ so that $\|[b,f]\|_{ql} = \limsup_{n \to \infty} \|[b, f_n]\| \geq \epsilon$, a contradiction. □

Note that, by the previous lemma, our old goal of proving

$$b \in C_u^*(X) \text{ if and only if } [b,f] = 0 \text{ for all } f \in VL_\infty(X)_1$$

is equivalent to our new goal

$$b \in C^*_u(X) \iff \forall \epsilon > 0 \text{ there exists an } L > 0 \text{ such that } b \in \mathrm{Commut}(L, \epsilon). \tag{6}$$

Definition 3.12. For a metric space (X, d), a cover $\mathcal{U} = \{U_i\}_{i \in I}$ is,

i) *uniformly bounded* if $\sup_{i \in I} \mathrm{diam}(U_i) < \infty$ where $\mathrm{diam}(U) := \sup_{x,y \in U}\{d(x, y)\}$ and has,

ii) *finite multiplicity* if there exists some M such that for each $x \in X$, at most M elements of $\mathcal{U}$ contain x.

Definition 3.13 (metric p-partition of unity, [10] definition 6.1)**.** For $p \in [0, \infty)$, a *metric p-partition of unity* on a metric space (X, d) is a collection

$$\{\phi_i : X \to [0, 1]\}_{i \in I} \text{ of functions on } X \text{ satisfying}$$

i) The cover $\{\mathrm{supp}(\phi_i)\}_{i \in I}$ is uniformly bounded and has finite multiplicity.

ii) For each $x \in X$, $\sum_{i \in I} \phi_i(x)^p = 1$.

Note that by this second condition viewing $\sum_{i \in I} \phi_i^2$ as an element of $C_b(X)$, $\sum_{i \in I} \phi_i^2$ is the identity in $\mathscr{B}(\ell^2(X))$.

Definition 3.14 (property A)**.** A metric p-partition of unity $\{\phi_i\}_{i \in I}$ has (r, ϵ)-variation if

$$\sum_{i \in I} |\phi_i(x) - \phi_i(y)|^p < \epsilon^p \text{ whenever } d(x, y) < r$$

The space X has property A if for any $p \in [1, \infty)$ and $r, \epsilon > 0$ there exists a metric p-partition of unity with (r, ϵ)-variation.

Remark 3.15. While this is not the original definition of property A given by Yu [13] definition 2.1; it is an equivalent definition by [12] Theorem 1.2.4, (6). Moreover, it is the definition that we shall need. Furthermore, by [2], property A implies the metric sparsification property below.

Definition 3.16 (metric sparsification property)**.** A metric space (X, d) has the *metric sparsification property* if for any $c \in [0, 1)$ there exists a nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ satisfying,

1. for any $M \in \mathbb{N}$, and

2. for any finite positive Borel measure μ on X;

there exists a Borel subset of X, $\Omega = \sqcup_{i \in I} \Omega_i$ such that,

i) $d(\Omega_i, \Omega_j) \geq M$, whenever $i \neq j$,

ii) $\mathrm{diam}(\Omega_i) \leq f(M)$ for every $i \in I$,

iii) $\mu(\Omega) \geq c\mu(X)$.

Note that the following lemmas still hold for a general $p \in (1, \infty)$. However, since we are concerned with $C_u^*(X)$ we only consider the case where $p = 2$. See [11] for a general treatment.

Lemma 3.17 ([11], Lemma 5.2)**.** *Suppose that X has the metric sparsification property. Then for any $\epsilon > 0$, $L > 0$, and $M > 0$, there exists an $s > 0$ such that for all $b \in \mathscr{B}(\ell^2(X))$ with $\|b\| \leq M$ and $b \in \mathrm{Commut}(L, \epsilon)$, there exists a unit vector $v \in \ell^2(X)$ with $\mathrm{diam}(\mathrm{supp}(v)) \leq s$, and satisfying:*

$$\|bv\| \geq \|b\| - 6\epsilon$$

To prove this we first make a claim to obtain an estimate. Then we proceed to the proof of the lemma. Let us fix $\epsilon, L, M > 0$ for the claim and proof of the lemma.

Claim 3.17.1. *Let $v \in \ell^2(X)$ have the form*

$$v = \sum_{j \in J} v_j \quad \textit{with} \quad d\left(\mathrm{supp}(v_j), \mathrm{supp}(v_i)\right) > \frac{4}{L}.$$

Then for $b \in \mathscr{B}(\ell^2(X))$ with $\|b\| \leq M$ and $b \in \mathrm{Commut}(L, \epsilon)$ we have that

$$\frac{\|bv\|}{\|v\|} \leq \sup_{j \in J} \frac{\|bv_j\|}{\|v_j\|} + 3\epsilon.$$

Proof of claim 3.17.1. First we need several estimates. Set $Y_j := \mathrm{supp}(v_j)$ and define

$$f_j(x) = \begin{cases} 1 & \text{if } \ x \in Y_j \\ 1 - L \cdot d(x, Y_j) & \text{if } \ 0 \leq d(x, Y_j) \leq \frac{1}{L} \\ 0 & \text{otherwise.} \end{cases}$$

Note that for each Y_j, $f_j \in (C_b(X))_1$ and f_j is a positive L-Lipschitz function such that

$$f\restriction_{Y_j}=1 \text{ and } \operatorname{supp}(f_j)\subseteq\{x\in X: d(x,Y_j)\leq \tfrac{1}{L}\}.$$

Moreover, the family $\{\operatorname{supp}(f_j): j\in J\}$ is $(2L^{-1})$-disjoint. Setting $f=\sum_{j\in J} f_j$ we have that $f\in C_b(X)_1$, f is L-Lipschitz and, $fv=v$. Furthermore, $(1-f)$ is L-Lipschitz since $\|(1-f(x))-(1-f(y))\|=\|f(x)-f(y)\|$. Combining this with the supposition that $b\in \operatorname{Commut}(L,\epsilon)$ we have

$$\|bv\| = \|fbv+bv-fbv-bv+bfv+bv-bfv\|$$

$$= \|fbv+((1-f)b-b(1-f))\,v+(bv-bfv)\|$$

$$\leq \|fbv\|+\|[(1-f),b]\|\,\|v\|\leq \|fbv\|+\epsilon\,\|v\|$$

Thus,

$$\|bv\|\leq \|fbv\|+\epsilon\,\|v\| \tag{7}$$

Note that $(f_j)_{j\in J}$ and the operator b satisfy the conditions of Corollary 3.8, and so $\left\|fbf-\sum_{j\in J} f_jbf_j\right\|\leq\epsilon$. Thus,

$$\|fbfv\|-\left\|\sum_{j\in J} f_jbf_jv\right\|\leq\left\|fbf-\sum_{j\in J} f_jbf_j\right\|\|v\|\leq\epsilon\,\|v\|$$

so that

$$\|fbfv\|\leq\left\|\sum_{j\in J} f_jbf_jv\right\|+\epsilon\,\|v\|\,.$$

Using this and that $f_jv=v_j$ & $fv=v$ we have that

$$\|fbv\|=\|fbfv\|\leq\left\|\sum_{j\in J} f_jbf_jv\right\|+\epsilon\,\|v\|=\left\|\sum_{j\in J} f_jbv_j\right\|+\epsilon\,\|v\|\,. \tag{8}$$

Since: $(1-f_j)$ is L-Lipschitz for all $j\in J$, $b\in\operatorname{Commut}(L,\epsilon)$ and,
$bv_j-bf_jv_j=0$;

$$\|(1-f_j)bv_j\|=\|bv_j-f_jbv_j-(bv_j-bf_jv_j)\|=\|[(1-f_j),b]v_j\|\leq\epsilon\,\|v_j\|\,. \tag{9}$$

Then using the triangle inequality (in the space $\ell^2\left(J,\ell^2(X)\right)$) and that $(f_j)_{j\in J}$

have mutually disjoint supports (cf.A.1), we have that

$$\left\|\sum_{j\in J} f_j b v_j\right\|_{\ell^2} = \left\|\sum_{j\in J} f_j b v_j\right\|_{\ell^2_J} = \left(\sum_{j\in J} \|f_j b v_j\|_{\ell^2}^2\right)^{1/2}$$

$$= \left(\sum_{j\in J} \|b v_j - (1-f_j) b v_j\|_{\ell^2}^2\right)^{1/2} \leq \left(\sum_{j\in J} \|b v_j\|_{\ell^2}^2\right)^{1/2} + \left(\sum_{j\in J} \|(1-f_j) b v_j\|_{\ell^2}^2\right)^{1/2}$$

$$\overset{(9)}{\leq} \left(\sum_{j\in J} \|b v_j\|_{\ell^2}^2\right)^{1/2} + \left(\sum_{j\in J} \epsilon^2 \|v_j\|_{\ell^2}^2\right)^{1/2} = \left(\sum_{j\in J} \|b v_j\|_{\ell^2}^2\right)^{1/2} + \epsilon \|v\|_{\ell^2}.$$

Thus,

$$\left\|\sum_{j\in J} f_j b v_j\right\|_{\ell^2} \leq \left(\sum_{j\in J} \|b v_j\|_{\ell^2}^2\right)^{1/2} + \epsilon \|v\|_{\ell^2}. \tag{10}$$

Observe that

$$\|bv\| \overset{(7)}{\leq} \|fbv\| + \epsilon \|v\| \overset{(8)}{\leq} \left\|\sum_{j\in J} f_j b v_j\right\| + 2\epsilon \|v\| \overset{(10)}{\leq} \left(\sum_{j\in J} \|b v_j\|^2\right)^{1/2} + 3\epsilon \|v\|$$

and so we obtain

$$(\|bv\| - 3\epsilon \|v\|)^2 \leq \sum_{j\in J} \|b v_j\|^2. \tag{11}$$

Now we are ready to prove the claim. Suppose for contradiction that the conclusion of the claim fails. Then, for any fixed $j \in J$, we have

$$\frac{\|bv\|}{\|v\|} > \sup_{i\in J} \frac{\|b v_i\|}{\|v_i\|} + 3\epsilon \geq \frac{\|b v_j\|}{\|v_j\|} + 3\epsilon$$

$$\implies \left(\frac{\|bv\|}{\|v\|} - \frac{3\epsilon \|v\|}{\|v\|}\right)^2 > \left(\frac{\|b v_j\|}{\|v_j\|}\right)^2$$

$$\implies \frac{(\|bv\| - 3\epsilon \|v\|)^2 \|v_j\|^2}{\|v\|^2} > \|b v_j\|^2. \tag{12}$$

However, that would mean that,

$$(\|bv\| - 3\epsilon \|v\|)^2 \overset{(11)}{\leq} \sum_{j\in J} \|b v_j\|^2 \overset{(12)}{<} \sum_{j\in J} \frac{(\|bv\| - 3\epsilon \|v\|)^2}{\|v\|^2} \|v_j\| = (\|bv\| - 3\epsilon \|v\|)^2$$

a contradiction. Therefore,

$$\frac{\|bv\|}{\|v\|} \leq \sup_{j\in J} \frac{\|bv_j\|}{\|v_j\|} + 3\epsilon.$$

□

Proof of Lemma 3.17. Recall that X has the metric sparsification property. Thus, for $c > 1-(\frac{\epsilon}{M})^2$ there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for any finite positive Borel measure μ on X, there exists a decomposition $\Omega = \sqcup_{j\in J}\Omega_j \subseteq X$ that satisfies:

i) $d(\Omega_j, \Omega_i) > 4/L$ whenever $j \neq i$,

ii) $\mathrm{diam}(\Omega_j) \leq f(4/L)$, and

iii) $\mu(\Omega) \geq c\mu(X)$.

Let $\omega \in \ell^2(X) \setminus \{0\}$. Define a measure μ_ω on X by $\mu_\omega(\{x\}) = |\omega(x)|^2$. Then we let $\Omega = \sqcup_{j\in J}\Omega_j$ be the associated decomposition. For any subset $Z \subseteq X$ we define $P_Z : \ell^2(X) \to \ell^2(X)$ by

$$(P_Z\xi)(x) = \begin{cases} \xi(x) & \text{if } x \in Z \\ 0 & \text{otherwise.} \end{cases}$$

Notice that P_Z is a contraction for all $Z \subseteq X$.

Next, for our measure μ_ω we make the following observations:

i)

$$\mu_\omega(X) = \sum_{x\in X} \mu_\omega(\{x\}) = \sum_{x\in X} |\omega(x)|^2 = \|\omega\|^2,$$

ii)

$$\|\omega - P_\Omega\omega\|^2 = \sum_{x\in X} |\omega(x) - P_\Omega\omega(x)|^2$$

$$= \sum_{x\in X\setminus\Omega} |\omega(x)|^2 = \sum_{x\in X\setminus\Omega} \mu_\omega(\{x\}) = \mu_\omega(X\setminus\Omega),$$

iii) and since $\mu_\omega(\Omega) \geq c\mu_\omega(X)$,

$$\|\omega\|^2 - \mu_\omega(\Omega) = \mu_\omega(X) - \mu_\omega(\Omega) = \mu_\omega(X\setminus\Omega) \leq (1-c)\|\omega\|^2 = \|\omega\|^2 - c\mu_\omega(X).$$

Putting this together we have that,

$$\|b\omega - bP_\Omega\omega\|^2 \le \|b\|^2 \|\omega - P_\Omega\omega\|^2 = \|b\|^2 \mu_\omega(x \setminus \Omega) \le M^2(1-c)\|\omega\|^2,$$

and so

$$\|bP_\Omega\omega\| \ge \|b\omega\| - M(1-c)^{1/2}\|\omega\| \tag{13}$$

where M was fixed before Claim 3.17.1. Note that, since $d(\Omega_j, \Omega_i) > 4/L$ whenever $j \neq i$, $P_\Omega\omega = \sum_{j\in J} P_{\Omega_j}\omega$ is a vector satisfying the conditions of Claim 3.17.1. Thus,

$$\sup_{j\in J} \frac{\|bP_{\Omega_j}\omega\|}{\|P_{\Omega_j}\omega\|} + 3\epsilon \ge \frac{\|bP_\Omega\omega\|}{\|P_\Omega\omega\|} \ge \frac{\|b\omega\| - M(1-c)^{1/2}\|\omega\|}{\|\omega\|} = \frac{\|b\omega\|}{\|\omega\|} - M(1-c)^{1/2}.$$

Now we take $\omega \in \ell^2(X) \setminus \{0\}$ such that $\frac{\|b\omega\|}{\|\omega\|} \ge \|b\| - \epsilon$ so that

$$\sup_{j\in J} \frac{\|bP_{\Omega_j}\omega\|}{\|P_{\Omega_j}\omega\|} \ge \|b\| - 4\epsilon - M(1-c)^{1/2}.$$

Moreover, Since $c > 1 - (\frac{\epsilon}{M})^2$ there exists a $j \in J$ such that

$$\frac{\|bP_{\Omega_j}\omega\|}{P_{\Omega_j}\omega} \le \|b\| - 6\epsilon$$

with $\operatorname{diam}(\operatorname{supp}(P_{\Omega_j}\omega)) \le f(4/L)$. Setting $s := f(4/L)$ and $v = \frac{P_{\Omega_j}\omega}{\|P_{\Omega_j}\omega\|}$, we complete the proof. □

Lemma 3.18. *Let $(\phi_j)_{j\in J}$ be a metric 2-partition of unity on X. Then for $b \in \mathscr{B}(\ell^2(X))$ we have that $\sum_{j\in J}\phi_j b\phi_j$ converges in the strong operator topology to an operator in $\mathscr{B}(\ell^2(X))$, and*

$$\left\|\sum_{j\in J}\phi_j b\phi_j\right\| \le \|b\|, \tag{14}$$

Proof. Since $\left\|\sum_{j\in J}\phi_j^2\right\| = 1$ by Lemma 3.1 $\sum_{j\in J}\phi_j b\phi_j$ converges strongly. Next, let $\mathcal{H} = \ell^2(X)$ and define

$$V : \mathcal{H} \to \ell^2(J, \mathcal{H}) \text{ by } V : \xi \mapsto (\phi_j\xi)_{j\in J} \text{ and}$$

$$V^* : \ell^2(J, \mathcal{H}) \to \mathcal{H} \text{ by } V^* : (\xi_j)_{j\in J} \mapsto \sum_{j\in J} \phi_j \xi_j.$$

We first show that V and V^* are well defined (in fact V is an isometry). Observe that,

$$\|V\xi\|^2_{\ell^2(J,\mathcal{H})} = \sum_{j\in J} \|\phi_j \xi\|^2_{\mathcal{H}} = \sum_{j\in J}\sum_{x\in X} |\phi_j(x)\xi(x)|^2 = \sum_{x\in X}\sum_{j\in J} \phi_j(x)\,|\xi(x)|^2 = \|\xi\|^2_{\mathcal{H}},$$

and that

$$\begin{aligned}
\|V^*(\xi_j)_{j\in J}\|_{\mathcal{H}} = \left\|\sum_{j\in J}\phi_j\xi_j\right\|_{\mathcal{H}} &= \left(\sum_{x\in X}\left|\sum_{j\in J}\phi_j(x)\xi_j(x)\right|^2\right)^{1/2} \\
&\le \left(\sum_{x\in X}\left(\sum_{j\in J}|\phi_j(x)\xi_j(x)|\right)^2\right)^{1/2} \\
&\overset{\text{C-S}}{\le} \left(\sum_{x\in X}\left(\left(\sum_{j\in J}|\phi_j(x)|^2\right)^{1/2}\left(\sum_{j\in J}|\xi_j(x)|^2\right)^{1/2}\right)^2\right)^{1/2} \\
&= \left(\sum_{x\in X}\sum_{j\in J}|\xi_j(x)|^2\right)^{1/2} = \|(\xi_j)_{j\in J}\|_{\ell^2(J,\mathcal{H})}.
\end{aligned}$$

Putting this together with the fact that

$$V^*V\xi = V^*(\phi_j\xi)_{j\in J} = \sum_{j\in J}\phi_j(\phi_j\xi) = \xi$$

we have that $\|V^*\| = 1 = \|V\|$. Define $(b_j)_{j\in J} \in \ell^\infty(J, \mathscr{B}(\mathcal{H}))$ by $b_j = b$ for all $j \in J$. Observe that,

$$V^*(b_j)_{j\in J}V\xi = V^*(b_j)_{j\in J}(\phi_j\xi)_{j\in J} = V^*(b\phi_j\xi)_{j\in J} = \sum_{j\in J}\phi_j b\phi_j\xi$$

Thus,

$$\left\|\sum_{j\in J}\phi_j b\phi_j\right\| = \|V^*(b_j)_{j\in J}V\| \le \|b\|$$

as was to be shown. □

Lemma 3.19 ([11], Section 5)**.** *Suppose that X has property A. Let $(\phi_j)_{j\in J}$ be a metric 2-partition of unity on X and let R be a uniform bound on $\{\mathrm{supp}(\phi_j)\}_{j\in J}$. Then for $b \in \mathscr{B}(\ell^2(X))$ we have that:*

i)

$$\sum_{j\in J} \phi_j b \phi_j \in \mathbb{C}_u^R[X], \text{ and}$$

ii)

$$\sum_{j\in J} \phi_j b \phi_j - b = \sum_{j\in J} \phi_j [b, \phi_j] \tag{15}$$

Proof. First observe that,

$$\left(\sum_{j\in J} \phi_j b \phi_j\right)_{xy} = \left\langle \sum_{j\in J} \phi_j b \phi_j \vartheta_x, \vartheta_y \right\rangle = \sum_{j\in J} \langle \phi_j b \phi_j \vartheta_x, \vartheta_y \rangle$$

$$= \sum_{j\in J} \langle b\phi_j \vartheta_x, \phi_j \vartheta_y \rangle = \sum_{j\in J} \phi_j(x)\phi_j(y) \langle b\vartheta_x, \vartheta_y \rangle = \sum_{j\in J} \phi_j(x)\phi_j(y) b_{xy} \tag{16}$$

Since R is a uniform bound on $\{\mathrm{supp}(\phi_j)\}_{j\in J}$, if $d(x,y) > R$, then $\phi_j(x)\phi_j(y) = 0$ for all $j \in J$ and so $\sum_{j\in J} \phi_j b \phi_j \in \mathbb{C}_u^R[X]$.

Lastly, we calculate

$$\sum_{j\in J} \phi_j [b, \phi_j] = \sum_{j\in J} \phi_j (b\phi_j - \phi_j b) = \sum_{j\in J} (\phi_j b \phi_j - \phi_j^2 b)$$

$$= \sum_{j\in J} \phi_j b \phi_j - \sum_{j\in J} \phi_j^2 b \overset{3.13}{=} \sum_{j\in J} \phi_j b \phi_j - b$$

as was to be shown. ∎

Lemma 3.20 ([11], Lemma 5.4)**.** *Let $b \in \mathrm{Commut}(L, \epsilon)$ for some $L, \epsilon > 0$. then for any metric 2-partition of unity $(\phi_j)_{j\in j}$, the operator $\sum_{j\in J} \phi_j [b, \phi_j] \in \mathrm{Commut}(L, 2\epsilon)$.*

Proof. Let f be an arbitrary L-Lipschitz function in $C_b(X)$. Note that, since $f, \phi_j \in C_b(X)$, f and ϕ_j commute for all $j \in J$. Thus,

$$\left\| \left[\sum_{j\in J} \phi_j [b, \phi_j], f \right] \right\| \overset{(15)}{=} \left\| \left[\left(\sum_{j\in J} \phi_j b \phi_j - b \right), f \right] \right\|$$

$$= \left\| \left(\sum_{j \in J} \phi_j b \phi_j - b \right) f - f \left(\sum_{j \in J} \phi_j b \phi_j - b \right) \right\|$$

$$= \left\| \left(\sum_{j \in J} \phi_j b \phi_j \right) f - bf - f \left(\sum_{j \in J} \phi_j b \phi_j \right) + fb \right\|$$

$$= \left\| \sum_{j \in J} \phi_j b f \phi_j - \sum_{j \in J} \phi_j f b \phi_j - [b, f] \right\|$$

$$= \left\| \sum_{j \in J} \phi_j [b, f] \phi_j - [b, f] \right\| \leq \left\| \sum_{j \in J} \phi_j [b, f] \phi_j \right\| + \|[b, f]\| \leq \epsilon + \epsilon = 2\epsilon$$

since $b \in \mathrm{Commut}(L, \epsilon)$. □

Theorem 3.21 ([11],Theorem 3.3)**.** *Let (X, d) be a metric space with bounded geometry having property A. Suppose that for $b \in \mathscr{B}(\ell^2(X))$ we have $[b, f] = 0$ for all $f \in VL_\infty(X)$. Then $b \in C_u^*(X)$.*

Proof. Let $\epsilon > 0$ be given and fix $b \in \mathscr{B}(\ell^2(X))$ such that $[b, f] = 0$ for all $f \in \mathrm{VL}_\infty(X)$. Set $M = \|b\|$, then by Lemma 3.11 there exists an $L > 0$ such that $b \in \mathrm{Commut}(L, \frac{\epsilon}{\max\{4M, 24\}})$. Next, applying Lemma 3.17 to $\epsilon/12, L, 2M$, we obtain an $s > 0$ such that for any operator $a \in \mathscr{B}(\ell^2(X)$ with $\|a\| \leq 2M$ and $a \in \mathrm{Commut}(L, \epsilon/12)$, there exists a unit vector $v \in \ell^2(X)$ with $\mathrm{diam}(\mathrm{supp}(v)) \leq s$, and satisfying $\|av\| \geq \|a\| - \epsilon/2$. Note that, since X has bounded geometry, $K := \sup_{z \in X} \left| B(z, s + \frac{1}{L}) \right| < \infty$.

Now, since X has property A, we may take a metric 2-partition of unity $(\phi_j)_{j \in J}$ with $(s + \frac{2}{L}, \frac{\epsilon}{4MK})$-variation. Take

$$b' = \sum_{j \in J} \phi_j b \phi_j.$$

Note that, by Lemmas 3.18 and 3.19, $b' \in \mathbb{C}_u^R[X]$ for some $R < \infty$ and $\|b'\| \leq \|b\| \leq M$. Let $a := b' - b$ so that $\|a\| \leq 2M$. Observe that, by (15) $a = \sum_{j \in J} \phi_j [b, \phi_j]$. Moreover, $a \in \mathrm{Commut}(L, \epsilon/12)$ by Lemma 3.20. Thus, there exists a unit vector $v \in \ell^2(X)$ with $\mathrm{diam}(\mathrm{supp}(v)) \leq s$ satisfying

$$\|av\| \geq \|a\| - \epsilon/2. \tag{17}$$

Take $F := \mathrm{supp}(v)$ and $G := \{x \in X : d(x, F) \leq 1/L\}$. Then we may find an L-Lipschitz function $f \in C_b(X)_1$ such that $f\restriction_{G^c}= 1$ and $f\restriction_F= 0$. Since $f\restriction_F= 0$, $fv = 0$, so that $(fa - af)v = fav = \chi_{G^c}av + \chi_G fav$. Thus,

$$\left\|\chi_{G^c}av\right\| = \left\|\chi_{G^c}[f,a]v\right\| \le \left\|\chi_{G^c}\right\| \|[a,f]\| \|v\| \le \epsilon/12$$

since $a \in \mathrm{Commut}(L, \epsilon/12)$ and v is a unit vector. Hence,

$$\|av\| \le \left\|\chi_G av + \chi_{G^c}av\right\| \le \left\|\chi_G av\right\| + \left\|\chi_{G^c}av\right\| \le \left\|\chi_G av\right\| + \epsilon/12 \tag{18}$$

Observe that,

$$a_{xy} = b'_{xy} - b_{xy} \overset{(16)}{=} \sum_{j\in J} \phi_j(x)\phi_j(y)b_{xy} - b_{xy}$$

$$\overset{3.13}{=} \sum_{j\in J} \phi_j(x)\phi_j(y)b_{xy} - \sum_{j\in J} \phi_j(x)^2 b_{xy} = \sum_{j\in J} \phi_j(x)\left(\phi_j(y) - \phi_j(x)\right) b_{xy}$$

Thus, for any $x \in G$ we have

$$|(av)(x)| = \left|\sum_{y\in F} a_{xy}v(y)\right| = \left|\sum_{y\in F}\sum_{j\in J} \phi_j(x)b_{xy}\left(\phi_j(y) - \phi_j(x)\right)v(y)\right| \tag{19}$$

Now, for $x \in G$ and each $y \in F$, $d(x,y) \le s + \frac{2}{L}$ so that

$$\left(\sum_{j\in J} |\phi_j(y) - \phi_j(x)|^2\right)^{1/2} \le \tfrac{\epsilon}{4MK} \tag{20}$$

since $(\phi_j)_{j\in J}$ has $(s + \frac{2}{L}, \frac{\epsilon}{4MK})$-variation. Hence,

$$\left|\sum_{j\in J} \phi_j(x)b_{xy}\left(\phi_j(y) - \phi_j(x)\right)\right| \le M \sum_{j\in J} \phi_j(x)\left|\phi_j(y) - \phi_j(x)\right|$$

$$\overset{C\text{-}S}{\le} M\left(\sum_{j\in J} \phi_j(x)^2\right)^{1/2} \cdot \left(\sum_{j\in J} |\phi_j(y) - \phi_j(x)|^2\right)^{1/2}$$

$$\overset{3.13}{=} M\left(\sum_{j\in J} |\phi_j(y) - \phi_j(x)|^2\right)^{1/2} \overset{(20)}{\le} M\frac{\epsilon}{4MK} = \frac{\epsilon}{4K} \tag{21}$$

Thus,

$$|(av)(x)| \overset{(19)}{\leq} \sum_{y\in F} \left| \sum_{j\in J} \phi_j(x) b_{xy} \left(\phi_j(y) - \phi_j(x)\right) \right| |v(y)|$$

$$\overset{(21)}{\leq} \frac{\epsilon}{4K} \sum_{y\in F} |v(y)| \overset{C\text{-}S}{\leq} \frac{\epsilon}{4K} |F|^{1/2} \cdot \left(\sum_{y\in F} |v(y)|^2 \right)^{1/2}$$

$$\leq \frac{\epsilon}{4K} \left(\sup_{y\in F} |B(y,s)| \right)^{1/2} \|v\| \leq \frac{\epsilon K^{1/2}}{4K}. \tag{22}$$

Combining (22) with (18) we obtain

$$\|av\| \overset{(18)}{\leq} \left\| \chi_G av \right\| + \frac{\epsilon}{12} = \left(\sum_{x\in G} |(av)(x)|^2 \right)^{1/2} + \frac{\epsilon}{12} \overset{(22)}{\leq} \frac{\epsilon K^{1/2}}{4K} \cdot K^{1/2} + \frac{\epsilon}{12} < \frac{\epsilon}{2}.$$

Therefore,

$$\|b' - b\| = \|a\| \overset{(17)}{\leq} \|av\| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and so $b \in C_u^*(X)$ □

4 Families of Operators and Conclusion

First we make a claim, then the results following the claim, due to Braga and Farah, will allow us to consider families of operators simultaneously.

Claim 4.0.1. *If $(V_\alpha)_{\alpha\in A}$ is a net of operators in $\mathbb{C}_u^R[X]$ converging weakly to some $V \in \mathscr{B}(\mathcal{H})$ then $V \in \mathbb{C}_u^R[X]$.*

Proof. First, since $V_\alpha \in \mathbb{C}_u^R[X]$

$$|\langle V_\alpha \vartheta_x, \vartheta_y \rangle| = 0 \quad \text{whenever} \quad d(x,y) > R \quad \text{for all} \quad \alpha \in A$$

Next, since this net weakly converges given any $\xi, \eta \in \ell^2(X)$ and $\epsilon > 0$ there exists an $\alpha_{\xi,\eta}$ such that,

$$|\langle (V_\alpha - V)\xi, \eta \rangle| < \epsilon \quad \text{whenever} \quad \alpha \geq \alpha_{\xi,\eta}.$$

Putting this together we have that,

$$\epsilon > |\langle (V_\alpha - V)\vartheta_x, \vartheta_y \rangle| = |\langle V\vartheta_x, \vartheta_y \rangle| \quad \text{whenever} \quad d(x,y) > R.$$

Thus, letting $\epsilon \to 0$, we have that $V_{xy} = 0$ whenever $d(x,y) > R$ and so $V \in \mathbb{C}_u^R[X]$ as was to be shown. □

Definition 4.1 (*ϵ-R-approximated*)**.** Given $\epsilon > 0$ and $R > 0$; an operator $T \in \mathscr{B}(\ell^2(X))$ can be ϵ-R-*approximated*, denoted $T \in_\epsilon \mathbb{C}_u^R[X]$, if there exists an $S \in \mathbb{C}_u^R[X]$ such that $\|T - S\| \leq \epsilon$.

Lemma 4.2 ([1], Lemma 4.6)**.** *Let (X,d) be a metric space with bounded geometry, and let $\epsilon > 0$. Let $T \in C_u^*(X)$ and let $P \in \ell^\infty(X)$ be a finite rank projection. Then,*

i) if T is ϵ-R-approximated then so is TP, and

ii) if TP is $(\epsilon+\gamma)$-R-approximated for all $\gamma > 0$, then TP is ϵ-R-approximated.

Proof. Since T is ϵ-R-approximated there exists a $V \in \mathbb{C}_u^R[X]$ such that $\|T - V\| < \epsilon$. Moreover, $\mathrm{supp}(VP) \subseteq \mathrm{supp}(V)$, so $VP \in \mathbb{C}_u^R[X]$ and $\|TP - VP\| \leq \|T - V\|\,\|P\| < \epsilon$ so *i)* holds.
For *ii)* we may choose an $V_n \in \mathbb{C}_u^R[X]$ such that $\|TP - V_n\| < \epsilon + \frac{1}{n}$ for every $n \in \mathbb{N}$. Define the sets

$$X' := \{x \in X : P\vartheta_x \neq 0\} \text{ and } X'' := \{x \in X : \exists x' \in X' \text{ such that } d(x,x') \leq R\}.$$

Let $l = |X'|$, and let $k = |X''|$. Since P is finite rank, and $V_n \in \mathbb{C}_u^R[X]$, both X' and X'' are finite sets. Moreover, for each $n \in \mathbb{N}$, V_nP can be naturally identified as an operator $S_nP : \ell^2(X') \to \ell^2(X'')$. Which, in turn can be identified with $M_{k,l}(\mathbb{C})$. Note that,

$$\|TP - V_nP\| \leq \|TP - V_n\|\,\|P\| \leq \epsilon + \frac{1}{n} \quad \text{so that,} \quad \|V_nP\| \leq \|T\| + \epsilon + 1$$

for all n. Thus, $(V_nP)_{n\in\mathbb{N}}$ is a bounded sequence in a finite dimensional space and so has a convergent subsequence which we also denote as $(V_nP)_{n\in\mathbb{N}}$. Let V be the limit of this subsequence. By Claim 4.0.1 $V \in \mathbb{C}_u^R[X]$; moreover,

$$\|TP - V\| = \lim_{n\to\infty} \|TP - V_nP\| < \lim_{n\to\infty} \epsilon + \frac{1}{n} = \epsilon.$$

Therefore, TP is ϵ-R-approximated. □

Lemma 4.3 ([1], Lemma 4.7)**.** *Let (X,d) be a metric space. Let $(P_j)_{j\in X}$ be an increasing net of finite rank projections in $\ell^\infty(X)$ converging strongly to the* *identity. Then for $\epsilon, R > 0$, if $T \in \mathscr{B}(\ell^2(X))$ cannot be ϵ-R-approximated then there exists a j_0 such that TP_j cannot be ϵ-R-approximated for all $j \geq j_0$.*

Proof. Suppose not. Then for all $j_0 \in J$ there exists $j \geq j_0$ such that TP_j is ϵ-R-approximated. However, since $(P_j)_{j\in X}$ is an increasing net of finite rank projections, $TP_jP_{j_0} = TP_{j_0}$ whenever $j \geq j_0$. Thus, by Lemma 4.2 TP_{j_0} is ϵ-R-approximated for all $j_0 \in X$. For every $j \in X$ fix a $S_j \in \mathbb{C}_u^R[X]$ such that $\|TP_j - S_j\| < \epsilon$. Note that, $\|S_j\| < \epsilon + \|T\|$ for all j so the net $(S_j)_{j\in J}$ is contained in a weakly compact subset of $\mathscr{B}(\ell^2(X))$. By passing to a subnet if necessary we may assume that $(S_j)_{j\in J}$ weakly converges to an operator S. Moreover, by Claim 4.0.1, $S \in \mathbb{C}_u^R[X]$ so that $\|T - S\| > \epsilon$ by assumption. Thus, there exists unit vectors ξ, η such that $|\langle (T - S)\xi, \eta\rangle| > \epsilon$ (cf.A.2). Then, there exists a j_0 such that $\|(TP_j - S_j)\xi\| > \epsilon$ whenever $j \geq j_0$ (cf.A.3). However, since $\|\xi\| = 1$,

$$\|(TP_j - S_j)\xi\| \leq \sup_{\|\zeta\|=1} \|(TP_j - S_j)\zeta\| = \|TP_j - S_j\|$$

which contradicts that S_j ϵ-R-approximates TP_j. ☐

Lemma 4.4 ([1], Lemma 4.8)**.** *Let (X, d) be a metric space, and let K be a compact subset of $C_u^*(X)$ in the norm topology. Then for every $\epsilon > 0$ there exists $R > 0$ such that every $T \in K$ can be ϵ-R-approximated.*

Proof. Given $\epsilon > 0$ for each $T \in K$ there exists some R_T such that T is ϵ-R_T-approximated. Thus, if K is finite we may take $R = \max\{R_T\}$. If K is infinite we build an open cover $B_{\epsilon/2}(T) := \{S \in K : \|S - T\| < \epsilon/2\}$. Since K is compact we may take a finite subcover. Then by the finite case there exists an R such that the center of each ball is $\epsilon/2$-R-approximated. Thus, by the triangle inequality, every element of K can be ϵ-R-approximated. ☐

Lemma 4.5. *Let $(T_\alpha)_{\alpha\in A}$ be a uniformly bounded net in $C_u^*(X)$ converging strongly to $T \in C_u^*(X)$. Then for any finite rank operator $k \in C_u^*(X)$, the net $(T_\alpha k)_{\alpha\in A}$ converges in norm.*

Proof. Suppose for contradiction that $(T_\alpha k)_{\alpha\in A}$ does not converge in norm. Let M be the uniform bound on $(T_\alpha k)_{\alpha\in A}$. Then there exists an $\epsilon > 0$ such that for all $j \in A$ there exists an $\alpha_j \geq j$ such that $\|T_{\alpha_j}k - Tk\| \geq \epsilon$. Hence, for all $j \in A$ there exists a $\xi_{\alpha_j} \in \ell^2(X)$, $\|\xi_{\alpha_j}\| = 1$ such that $\|(T_{\alpha_j}k - Tk)\xi_{\alpha_j}\| > \frac{\epsilon}{2}$.

Next, since k is finite rank, the image of the closed unit ball of $\ell^2(X)$ is compact under k. Thus, the net $(k\xi_{\alpha_j})_{j\in A}$ has a convergent subnet, $k\xi_\beta \to \eta$. Then, since $(T_\alpha)_{\alpha\in A}$ converges strongly there exists an α_η such that

$$\|(T_\alpha - T)\eta\| < \tfrac{\epsilon}{4} \text{ whenever } \alpha \geq \alpha_\eta.$$

Moreover, there exists a β_0 such that

$$\|k\xi_\beta - \eta\| < \tfrac{\epsilon}{4(M+\|T\|)} \text{ whenever } \beta \geq \beta_0.$$

Taking $\beta \geq \alpha_\eta,\ \beta \geq \beta_0$ we have

$$\|(T_\beta k - Tk)\xi_\beta\| = \|(T_\beta - T)(k\xi_\beta - \eta + \eta)\|$$

$$= \|(T_\beta - T)(\eta) + (T_\beta - T)(k\xi_\beta - \eta)\|$$

$$\leq \|(T_\beta - T)(\eta)\| + (\|T_\beta\| + \|T\|)\,\|k\xi_\beta - \eta\| \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}$$

contradicting that $\|(T_\beta k - Tk)\xi_\beta\| > \frac{\epsilon}{2}$. □

It will be convenient in what follows to denote a set indexed by X, of elements $\lambda_x \in \mathbb{D}$, by $\overline{\lambda} \in \mathbb{D}^X$ where $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$.

Lemma 4.6. *Suppose that $(T_j)_{j\in X}$ is a family of finite rank operators in $C_u^*(X)$ such that for every $\overline{\lambda} \in \mathbb{D}^X$ the series $\sum_{j\in X} \lambda_j T_j$ converges strongly to an operator $T_{\overline{\lambda}} \in C_u^*(X)$. Moreover, suppose that $\left\|\sum_{j\in F} \lambda_j T_j\right\|$ is uniformly bounded for all finite $F \subseteq X$. Then, for any finite rank projection $P \in C_u^*(X)$ and any $\delta > 0$ there exists a finite set I such that for all finite $K \supseteq I$ and all $\overline{\lambda} \in \mathbb{D}^X$ we have that*

$$\left\|\sum_{j\in X\setminus K} \lambda_j T_j P\right\| < \delta$$

Proof. First, let $T_j P = x_j$ for all $j \in X$. Then by Lemma 4.5 $\sum_{j\in X} \lambda_j x_j$ converges in norm for all $\overline{\lambda} \in \mathbb{D}^X$. Suppose for contradiction that the conclusion of the lemma fails. Then,

$$(\exists\delta > 0)(\forall F_0 \overset{\text{finite}}{\subset} X)(\exists F \overset{\text{finite}}{\supseteq} F_0)(\exists\overline{\lambda} \in \mathbb{D}^X) \text{ such that } \left\|\sum_{j\in X\setminus F} \lambda_j x_j\right\| \geq \delta$$

Then, we may inductively build a chain of finite subsets

$$F_1 \subseteq H_1 \subseteq F_2 \subseteq H_2 \subseteq \dots \text{ such that } \left\| \sum_{j \in H_k \setminus F_k} \lambda_j^{(k)} x_j \right\| > \frac{\delta}{2}.$$

Indeed, let δ be as above and let F_0 be arbitrary. By supposition there exists a $F_1 \supseteq F_0$ and a $\overline{\lambda}^{(1)} \in \mathbb{D}^X$ such that

$$\left\| \sum_{j \in X \setminus F_1} \lambda_j^{(1)} x_j \right\| \geq \delta.$$

Since $\sum_{j \in X} \lambda_j^{(1)} x_j$ converges there exists a finite set E_1 such that for all finite $H \supseteq E_1$ we have

$$\left\| \sum_{j \in X \setminus H} \lambda_j^{(1)} x_j \right\| < \frac{\delta}{2}.$$

Take $H_1 = F_1 \cup E_1$; then,

$$\left\| \sum_{j \in X \setminus H_1} \lambda_j^{(1)} x_j + \sum_{j \in H_1 \setminus F_1} \lambda_j^{(1)} x_j \right\| = \left\| \sum_{j \in X \setminus F_1} \lambda_j^{(1)} x_j \right\| \geq \delta$$

so that

$$\left\| \sum_{j \in H_1 \setminus F_1} \lambda_j^{(1)} x_j \right\| > \frac{\delta}{2}.$$

Assume now that we have carried out the construction to step n. By supposition there exists a $F_{n+1} \supseteq H_n$ and a $\overline{\lambda}^{(n+1)} \in \mathbb{D}^X$ such that

$$\left\| \sum_{j \in X \setminus F_{n+1}} \lambda_j^{(n+1)} x_j \right\| \geq \delta.$$

Since $\sum_{j \in X} \lambda_j^{(n+1)} x_j$ converges there exists a finite set E_{n+1} such that for all finite $H \supseteq E_{n+1}$ we have

$$\left\| \sum_{j \in X \setminus H} \lambda_j^{(n+1)} x_j \right\| < \frac{\delta}{2}.$$

Take $H_{n+1} = F_{n+1} \cup E_{n+1}$; then,

$$\left\| \sum_{j \in X \setminus H_1} \lambda_j^{(n+1)} x_j + \sum_{j \in H_{n+1} \setminus F_{n+1}} \lambda_j^{(n+1)} x_j \right\| = \left\| \sum_{j \in X \setminus F_{n+1}} \lambda_j^{(n+1)} x_j \right\| \geq \delta$$

so that

$$\left\| \sum_{j \in H_{n+1} \setminus F_{n+1}} \lambda_j^{(n+1)} x_j \right\| > \frac{\delta}{2}.$$

Note that

$$H_k = F_k \cup E_k \quad \text{and} \quad F_k \supseteq H_{k-1} \supseteq F_{k-1} \ldots.$$

Next, define an element $\overline{\lambda} \in \mathbb{D}^X$ by

$$\lambda_j = \begin{cases} \lambda_j^{(k)} & \text{if } j \in H_k \setminus F_k \\ 0 & \text{otherwise} \end{cases}.$$

Since $\sum_{j \in X} \lambda_j x_j$ converges there exists a finite set F_λ such that for all finite sets $H \supseteq F \supseteq F_\lambda$ we have

$$\left\| \sum_{j \in H \setminus F} \lambda_j x_j \right\| < \frac{\delta}{4}.$$

Now we claim that there exists an $m \in \mathbb{N}$ such that $F_\lambda \cap (H_m \setminus F_m) = \emptyset$. Once this is shown we will have a contradiction by taking $F = F_\lambda$ and $H = F_\lambda \cup (H_m \setminus F_m)$ since

$$\left\| \sum_{j \in H \setminus F} \lambda_j x_j \right\| = \left\| \sum_{j \in H_m \setminus F_m} \lambda_j x_j \right\| \geq \frac{\delta}{2} \quad .$$

Thus, we need only show that there exists an $m \in \mathbb{N}$ such that $F_\lambda \cap (H_m \setminus F_m) = \emptyset$. Note that, since $\left\| \sum_{j \in H_m \setminus F_m} \lambda_j x_j \right\| \geq \frac{\delta}{2}$, $(H_k \setminus F_k) \neq \emptyset$ for all $k \in \mathbb{N}$. Moreover, since F_λ is finite $|F_\lambda| = n$ for some $n \in \mathbb{N}$. Hence, there are at most n of the $(H_k \setminus F_k)$'s such that $(H_k \setminus F_k) \cap F_\lambda \neq \emptyset$ and so we are done.

□

Lemma 4.7 ([1] Lemma 4.9)**.** *Let (X, d) be a metric space with bounded geometry. Suppose that $(T_j)_{j \in X}$ is a family of finite rank operators in $C_u^*(X)$ such that for every $\overline{\lambda} \in \mathbb{D}^X$ the series $\sum_{j \in X} \lambda_j T_j$ converges strongly to an operator $T_{\overline{\lambda}} \in C_u^*(X)$. Moreover, suppose that $\left\| \sum_{j \in F} \lambda_j T_j \right\|$ is uniformly bounded for* *all finite $F \subseteq X$. Then for every $\epsilon > 0$ there exists $R > 0$ such that $T_{\overline{\lambda}}$ can be ϵ-R-approximated for all $\overline{\lambda} \in \mathbb{D}^X$. In symbols:*

$$(\forall \epsilon > 0)(\exists R > 0)(\forall \overline{\lambda} \in \mathbb{D}^X) \text{ such that } T_{\overline{\lambda}} \textit{ is } \epsilon\text{-}R\textit{-approximated.}$$

Before we prove this we fix some notation. For each finite $I \subset X$ let

$$\mathcal{Z}_I := \{\overline{\lambda} \in \mathbb{D}^X : \forall j \in I,\ \lambda_j = 0\} \text{ and } \mathcal{Y}_I := \{\overline{\lambda} \in \mathbb{D}^X : \forall j \notin I,\ \lambda_j = 0\}$$

The structure of the proof is as follows. We assume for contradiction that the conclusion of the lemma fails. If the conclusion of the lemma fails it implies a stronger condition. We will use the stronger condition to build a cover of $\mathbb{D}^X$ that consists of closed sets having empty interiors. Then, by a cardinality argument, we show that this is a contradiction. To do this we make three claims, then we use the claims to demonstrate the contradiction.

Claim 4.7.1. *If there exists an $\epsilon > 0$ such that for all $R > 0$ there exists a $\overline{\lambda} \in \mathbb{D}^X$ such that $T_{\overline{\lambda}}$ is not ϵ-R-approximated, then there exists an $\epsilon' > 0$ such that for all $R > 0$ and all finite $I \subset X$ there exists a $\overline{r} \in \mathcal{Z}_I$ such that $T_{\overline{r}}$ is not ϵ'-R-approximated. In symbols:*

$$(\exists \epsilon > 0)(\forall R > 0)(\exists \overline{\lambda} \in \mathbb{D}^X) \text{ such that } T_{\overline{\lambda}} \textit{ is not } \epsilon\text{-}R\textit{-approximated} \implies$$

$$(\exists \epsilon' > 0)(\forall R > 0)(\forall I \overset{\textit{finite}}{\subset} X)(\exists \overline{r} \in \mathcal{Z}_I) \text{ such that } T_{\overline{r}} \textit{ is not } \epsilon'\text{-}R\textit{-approximated.}$$

Proof of claim 4.7.1. Suppose not. Then for all $\epsilon' > 0$ there exists an $R > 0$ and a finite $I \subset X$ such that for all $\overline{r} \in \mathcal{Z}_I$, $T_{\overline{r}}$ is ϵ'-R-approximated. In symbols:

$$(\forall \epsilon' > 0)(\exists R > 0)(\exists I \overset{\textit{finite}}{\subset} X)(\forall \overline{r} \in \mathcal{Z}_I) \text{ such that } T_{\overline{r}} \text{ is } \epsilon'\text{-}R\text{-approximated}$$

Take ϵ such that it satisfies the premise of our claim. Fix $\epsilon' = \epsilon/2$, then for the corresponding R and I we have that for all $\overline{r} \in \mathcal{Z}_I$, $T_{\overline{r}}$ is ϵ'-R-approximated. Note that,

$$\text{for any } \overline{s} \in \mathcal{Y}_I \text{ the operator } T_{\overline{s}} = \sum_{j \in X} s_j T_j \in C_u^*(X),$$

since it is equal to a finite linear combination of elements in $C_u^*(X)$. Moreover, $\mathcal{Y}_I$ is homeomorphic to $\mathbb{D}^I$ (which is compact) and the map $\overline{s} \mapsto T_{\overline{s}}$ is norm continuous for all $\overline{s} \in \mathcal{Y}_I$. Thus, the set $\{T_{\overline{s}} : \overline{s} \in \mathcal{Y}_I\}$ is compact. Then, by

Lemma 4.4, there exists an S such that $T_{\overline{s}}$ can be ϵ'-S-approximated for all $\overline{s} \in \mathcal{Y}_I$. Without loss of generality we may assume that $R \leq S$ (otherwise take S to be R). Now,

$$\text{for any } \overline{\lambda} \in \mathbb{D}^X,\ T_{\overline{\lambda}} = T_{\overline{r}} + T_{\overline{s}}$$

for some $\overline{r} \in \mathcal{Z}_I$ and some $\overline{s} \in \mathcal{Y}_I$. Hence, by the triangle inequality, $T_{\overline{\lambda}}$ can be $2\epsilon'$-S-approximated; that is, $T_{\overline{\lambda}}$ can be ϵ-S-approximated, which contradicts our premise. □

Claim 4.7.2. *If $T_{\overline{\lambda}}$ is ϵ-R-approximated but $T_{\overline{\theta}}$ is not 2ϵ-R-approximated. Then, $T_{\overline{\lambda}} + T_{\overline{\theta}}$ is not ϵ-R-approximated.*

Proof of claim 4.7.2. Suppose for contradiction that $T_{\overline{\lambda}}+T_{\overline{\theta}}$ is ϵ-R-approximated. Then there exists a $G \in \mathbb{C}_u^R[X]$ such that $\left\|T_{\overline{\lambda}} + T_{\overline{\theta}} - G\right\| < \epsilon$. Additionally, since $T_{\overline{\lambda}}$ is ϵ-R-approximated there exists a $L \in \mathbb{C}_u^R[X]$ such that $\left\|T_{\overline{\lambda}} - L\right\| < \epsilon$ Take $K = G - L$ so that $G = K + L$ and note that $K \in \mathbb{C}_u^R[X]$. Observe that

$$\left\|T_{\overline{\theta}} - K\right\| - \left\|T_{\overline{\lambda}} - L\right\| \leq \left\|T_{\overline{\theta}} - K + T_{\overline{\lambda}} - L\right\| = \left\|T_{\overline{\lambda}} + T_{\overline{\theta}} - G\right\| < \epsilon$$

Which implies that

$$\left\|T_{\overline{\theta}} - K\right\| < 2\epsilon$$

a contradiction. □

Claim 4.7.3. *If the conclusion of Lemma 4.7 fails; then for each $R > 0$ the set*

$$U_R := \left\{\overline{\lambda} \in \mathbb{D}^X : T_{\overline{\lambda}} \textit{ is } \epsilon\text{-}R\textit{-approximated}\right\},$$

is closed and has empty interior.

Proof of claim 4.7.3. Fix $\epsilon = \epsilon'/2$ where ϵ' is given by claim 4.7.1 First we show that each U_R is closed. Suppose for contradiction that there exists an R such that U_R is not closed. Choose a $\overline{\lambda} \in \overline{U_R} \setminus \left(U_R \cup \operatorname{Int}\left(\overline{U_R}\right)\right)$, such that $T_{\overline{\lambda}}$ is not ϵ-R-approximated. Then by Lemma 4.3, there exists a finite rank projection $P \in \ell^\infty$ such that $T_{\overline{\lambda}}P$ is not ϵ-R-approximated. Let $\delta > 0$ be fixed but arbitrary. By Lemma 4.6 there exists a finite set I such that $\left\|T_{\overline{\theta}}P\right\| < \delta$ whenever $\overline{\theta} \in \mathcal{Z}_I$. For this I we may write our $\overline{\lambda}$ as $\overline{\lambda} = \overline{\lambda}_I + \overline{\lambda}_\infty$ for some $\overline{\lambda}_I \in \mathcal{Y}_I$ and $\overline{\lambda}_\infty \in \mathcal{Z}_I$. Since U_R is not closed, there exists a nonempty subset of X, say H, such that $\{\eta_h \in \mathbb{D} : T_{\overline{\eta}} \in_\epsilon \mathbb{C}_u^R[X]\}$ is not closed whenever $h \in H$.

Thus, since $\overline{\lambda} \in \overline{U_R}$, for every $h \in H \cap I$ there exists a $\overline{\theta} \in U_R$ such that

$$0 \leq |\lambda_h - \theta_h| < {}^{\delta}\!\big/_{\sup_{j\in I} \|T_j\| \, |I|}.$$

Then, for this $\overline{\theta}$ we have

$$\left\|T_{\overline{\lambda}_I} - T_{\overline{\theta}_I}\right\| \leq \sup_{j\in I} \|T_j\| \sum_{j\in I} |\lambda_j - \theta_j| < \delta.$$

Moreover, since $T_{\overline{\theta}}$ is ϵ-R-approximated, by Lemma 4.2 there exists a $V_{\overline{\theta}} \in \mathbb{C}_u^R[X]$ such that $\left\|T_{\overline{\theta}}P - V_{\overline{\theta}}\right\| < \epsilon$. Observe that,

$$\left\|T_{\overline{\lambda}}P - V_{\overline{\theta}}\right\| = \left\|T_{\overline{\lambda}}P - T_{\overline{\theta}}P + T_{\overline{\theta}}P - V_{\overline{\theta}}\right\|$$

$$= \left\|T_{\overline{\theta}}P - V_{\overline{\theta}} + T_{\overline{\lambda}_I}P - T_{\overline{\theta}_I}P - T_{\overline{\theta}_\infty}P + T_{\overline{\lambda}_\infty}P\right\|$$

$$\leq \left\|T_{\overline{\theta}}P - V_{\overline{\theta}}\right\| + \left\|T_{\overline{\lambda}_I}P - T_{\overline{\theta}_I}P\right\| + \left\|T_{\overline{\theta}_\infty}P\right\| + \left\|T_{\overline{\lambda}_\infty}P\right\| < \epsilon + 3\delta$$

Thus, $T_{\overline{\lambda}}P$ is $(\epsilon + 3\delta)$-R-approximated. Then, by Lemma 4.2*(ii)*, this would mean that $T_{\overline{\lambda}}P$ is ϵ-R-approximated since δ was arbitrary. However, this contradicts that $T_{\overline{\lambda}}P$ is not ϵ-R-approximated for this P. Hence, U_R is closed for each R.

Now we show that U_R has empty interior for each R. Choose a $\overline{\lambda} \in \mathbb{D}^X$ and let $I \subset X$ be a finite, fixed but arbitrary subset. For this I we may write $\overline{\lambda} = \overline{\lambda}_I + \overline{\lambda}_\infty$ as before. Note that, there exists an $R' \geq R$ such that $T_{\overline{\lambda}_I}$ can be ϵ-R'-approximated. Recall that we fixed $\epsilon = \epsilon'/2$ where ϵ' is given by claim 4.7.1. So, there exists a $\overline{\theta}_\infty \in \mathcal{Z}_I$ such that $T_{\overline{\theta}_\infty}$ is not 2ϵ-R'-approximated by 4.7.1. Thus, by Claim 4.7.2 $T_{\overline{\lambda}_I} + T_{\overline{\theta}_\infty}$ is not ϵ-R'-approximated. Since $R \leq R'$ we have that $\overline{\lambda}_I + \overline{\theta}_\infty \notin U_R$; and since I was arbitrary this shows that U_R has empty interior. □

Proof of Lemma 4.7. Observe that, since for every element $\overline{\lambda} \in \mathbb{D}^X$, $T_{\overline{\lambda}} \in C_u^*(X)$ by supposition, every $\overline{\lambda} \in U_R$ for some R. Thus,

$$\mathbb{D}^X = \bigcup_{R\in\mathbb{N}} U_R.$$

By claim 4.7.3 if the conclusion of the lemma fails each U_R is closed and has empty interior. That would mean that each U_R is nowhere dense. Thus, we may cover $\mathbb{D}^X$ by a countable union of nowhere dense subsets. However, $\mathbb{D}^X$ is a nonempty complete metric space and so by the Baire category theorem $\mathbb{D}^X$ cannot be covered by a countable union of nowhere dense subsets, a contradiction. $\square$

The last piece we will need follows from [8], chapter 11.

Lemma 4.8. *Let (X, d) be a metric space with bounded geometry having property A. Let $(T_k)_{k\in K}$ be a family of operators. If there exists an $R > 0$ such that T_k can be ϵ-R-approximated for all $k \in K$ then there exists an S (dependent on ϵ and R) such that for all $k \in K$ there exists a $W_k \in \mathbb{C}_u^S[X]$ with $\|W_k - T_k\| \leq 3\epsilon$ and $|(W_k)_{xy}| \leq |(T_k)_{xy}|$ for all $x, y \in X$.*

Proof. Since X has property A, by [12] Theorem 1.2.4 (8) for each $R > 0$, $n \in \mathbb{N}$ there is a normalized, finite propagation, symmetric, positive type kernel

$$u_n : X \times X \to \mathbb{R} \text{ such that } 1 - \frac{1}{2n^2} < u_n(x, y) \leq 1 \text{ whenever } d(x, y) \leq R.$$

Moreover, by [8] Theorem 11.15, u_n is realized by a map $\varphi : X \to \mathcal{H}$ (where $\mathcal{H}$ is a real Hilbert space) so that $u_n(x, y) = \langle \varphi(x), \varphi(y)\rangle$. Thus, combining the normalization condition $u_n(x, x) = 1$ and the Cauchy-Schwartz inequality we have that $|u_n(x, y)| \leq 1$ for all $x, y \in X$. Next, by [8] Lemma 11.17 and Corollary 11.18, each u_n induces a unique unital completely positive contractive map

$$U_n : C_u^*(X) \to C_u^*(X) \text{ such that } \langle (U_n T)\vartheta_x, \vartheta_y\rangle = u_n(x, y) \langle T\vartheta_x, \vartheta_y\rangle .$$

and the sequence $\{U_n\}_{n\in\mathbb{N}}$ converges pointwise to the identity. Note that, $|(U_n T)_{xy}| \leq |T_{xy}|$ for all $x, y \in X$ and all n, by the normalization condition.

Next, since each T_k in our family of operators can be ϵ-R-approximated, for each T_k there exists a V_k such that $\|T_k - V_k\| < \epsilon$ and $V_k \in \mathbb{C}_u^R[X]$. Note that $u_n \to 1$ uniformly as $n \to \infty$ on the set $\{(x, y) : d(x, y) \leq R\}$. Hence, $U_n V_k \to V_k$ uniformly for all k and so there exits an n_0 such that $\|U_n V_k - V_k\| < \epsilon$ for all k whenever $n \geq n_0$. Fix an $n > n_0$ and define

$$W_k := U_n T_k.$$

Observe that, since U_n is contractive, we have

$$\|W_k - T_k\| = \|U_n T_k - U_n V_k + U_n V_k - V_k + V_k - T_k\|$$

$$\leq \|U_n\|_{op}\|T_k - V_k\| + \|U_nV_k - V_k\| + \|T_k - V_k\| < 3\epsilon$$

as was to be shown. □

Now we are ready to prove the main theorem.

Theorem 4.9. *Let (X, d) be a metric space with bounded geometry having property A. Then all bounded derivations on $C_u^*(X)$ are inner.*

Proof. By theorem 2.9 every bounded derivation on $C_u^*(X)$ is spatially implemented. Define $B = \{b \in \mathscr{B}(\ell^2(X)) : [b, a] \in C_u^*(X) \text{ for all } a \in C_u^*(X)\}$; that is, B is the set of all elements in $\mathscr{B}(\ell^2(X))$ that implement a derivation. Let $b \in B$ be fixed but arbitrary, and define p_j to be the rank one projection onto the j^{th} coordinate of $\ell^2(X)$.

For $\overline{\lambda} \in \mathbb{D}^X$ define $g_{\overline{\lambda}} = \sum_{j\in X} \lambda_j p_j$ so that $g_{\overline{\lambda}} \in (\ell^\infty(X))_1$ for all $\overline{\lambda} \in \mathbb{D}^X$. Let $\xi \in \ell^2(X)$ be fixed but arbitrary and $\epsilon > 0$ be given. Since $\xi \in \ell^2(X)$ there exists a finite set F_1 such that

$$\left(\sum_{j\in X\setminus F} |\xi_j|^2\right)^{1/2} < \frac{\epsilon}{2\|b\|} \text{ whenever } F_1 \subseteq F$$

Additionally, $b\xi \in \ell^2(X)$ so there exists a finite set F_2 such that

$$\left(\sum_{j\in X\setminus F} |(b\xi)_j|^2\right)^{1/2} < \frac{\epsilon}{2} \text{ whenever } F_2 \subseteq F.$$

Thus,

$$\left\|[b, g_{\overline{\lambda}}]\xi - \sum_{j\in F} \lambda_j [b, p_j]\xi\right\|_{\ell^2} = \left\|[b, \sum_{j\in X} \lambda_j p_j]\xi - [b, \sum_{j\in F} \lambda_j p_j]\xi\right\|_{\ell^2}$$

$$= \left\|b\left(\sum_{j\in X} \lambda_j p_j\right)\xi - \left(\sum_{j\in X} \lambda_j p_j\right)b\xi - b\left(\sum_{j\in F} \lambda_j p_j\right)\xi + \left(\sum_{j\in f} \lambda_j p_j\right)b\xi\right\|_{\ell^2}$$

$$= \left\|b\left(\sum_{j\in X\setminus F} \lambda_j p_j\right)\xi - \left(\sum_{j\in X\setminus F} \lambda_j p_j\right)b\xi\right\|_{\ell^2}$$

$$\leq \|b\| \left\| \sum_{j\in X\setminus F} \lambda_j p_j \xi \right\|_{\ell^2} + \left\| \sum_{j\in X\setminus F} \lambda_j p_j b\xi \right\|_{\ell^2}$$

$$= \|b\| \left(\sum_{j\in X\setminus F} |\lambda_j \xi_j|^2 \right)^{1/2} + \left(\sum_{j\in X\setminus F} |\lambda_j (b\xi)_j|^2 \right)^{1/2}$$

$$\leq \|b\| \left(\sum_{j\in X\setminus F} |\xi_j|^2 \right)^{1/2} + \left(\sum_{j\in X\setminus F} |(b\xi)_j|^2 \right)^{1/2} < \|b\| \frac{\epsilon}{2\|b\|} + \frac{\epsilon}{2} = \epsilon$$

whenever $F \supseteq F_1 \cup F_2$. Hence, $\sum_{j\in X} \lambda_j [b, p_j]$ converges strongly to $[b, g_{\overline{\lambda}}] \in C^*_u(X)$ for every $\overline{\lambda} \in \mathbb{D}^X$ and so $\{[b, p_j]\}_{j\in X}$ is a family of operators satisfying the conditions of lemma 4.7. Moreover, every $g \in (\ell^\infty(X))_1$ is given by $g_{\overline{\lambda}}$ for some $\overline{\lambda} \in \mathbb{D}^X$.

Hence, for every $\epsilon > 0$ there exists an $R > 0$ such that $[b, g]$ can be ϵ-R-approximated for all $g \in \ell^\infty(X)$. Then by 4.8 there exists an $S > 0$ such that for all $g \in \ell^\infty(X)$ there exists $W_g \in \mathbb{C}^S_u[X]$ with $\|W_g - [b, g]\| \leq 3\epsilon$ and $|(W_g)_{xy}| \leq \left|[b, g]_{xy}\right|$. Define $[b, g]^S$ to be the operator

$$[b, g]^S_{xy} = \begin{cases} [b, g]_{xy} & \text{if } d(x, y) \leq S \\ 0 & otherwise \end{cases}$$

so that $|(W_g)_{xy}| \leq \left|[b, g]^S_{xy}\right|$. Observe that,

$$\limsup_{\mathrm{Lip}(g)\to 0} \sup_{d(x,y)\leq S} |(W_g)_{xy}| \leq \limsup_{\mathrm{Lip}(g)\to 0} \sup_{d(x,y)\leq S} \left|[b, g]^S_{xy}\right|$$

$$\leq \|b\| \limsup_{\mathrm{Lip}(g)\to 0} \sup_{d(x,y)\leq S} \left|g(x) - g(y)\right| \leq \limsup_{\mathrm{Lip}(g)\to 0} \mathrm{Lip}(g) \sup_{d(x,y)\leq S} d(x, y)$$

$$\leq \limsup_{\mathrm{Lip}(g)\to 0} \mathrm{Lip}(g)S = 0.$$

Thus, $|(W_g)_{xy}| \to 0$ uniformly as $\mathrm{Lip}(g) \to 0$ so that $\lim_{\mathrm{Lip}(g)\to 0} \|W_g\| = 0$ by Lemma 2.5. Since,

$$\|[b, g]\| - \|W_g\| \leq \|[b, g] - W_g\| \overset{(4.8)}{\leq} 3\epsilon$$

$$\implies \limsup_{\mathrm{Lip}(g)\to 0} \|[b, g]\| \leq \limsup_{\mathrm{Lip}(g)\to 0} \|W_g\| + 3\epsilon,$$

letting $\epsilon \to 0$ we have that $\|[b,g]\| \to 0$ as $\mathrm{Lip}(g) \to 0$. Thus, given any $f \in VL_\infty$ we have that $\|[b,f]\|_{ql} = \limsup_{n\to\infty} \|[b,f_n]\| = 0$, and so $b \in C_u^*(X)$ Therefore, since b was an arbitrary element in B, $B = C_u^*(X)$ and so all bounded derivations on $C_u^*(X)$ are inner. $\square$

A Functional details

Proposition A.1. Let $b \in \mathscr{B}(\ell^2(X))$, $(f_j)_{j\in J}$ and $(v_j)_{j\in J}$ be as in Claim 3.17.1. Consider the space $\ell^2(J,\ell^2(X))$; then,

$$\left\|\sum_{j\in J} f_j b v_j\right\|_{\ell^2(J,\ell^2(X))} = \left\|\sum_{j\in J} f_j b v_j\right\|_{\ell^2(X)} \quad \text{and} \quad \left\|\sum_{j\in J} v_j\right\|_{\ell^2(J,\ell^2(X))} = \|v\|_{\ell^2(X)}$$

Proof. We prove the first equality. The second is proved similarly. First, since the supports of the f_i's are mutually disjoint for each $x \in X$ there exists precisely one $j \in J$ such that

$$\left(\sum_{i\in J} f_i b v_i\right)(x) = (f_j b v_j)(x)$$

and if $x \notin \mathrm{supp}(f_i)$ then $(f_i b v_i)_x = 0$. Thus,

$$\left\|\sum_{j\in J} f_j b v_j\right\|_{\ell^2(X)} = \left(\sum_{x\in X}\left|\left(\sum_{j\in J} f_j b v_j\right)(x)\right|^2\right)^{1/2} = \left(\sum_{i\in J}\sum_{x\in\mathrm{supp}(f_i)} |(f_i b v_i)(x)|^2\right)^{1/2}$$

$$= \left(\sum_{i\in J}\sum_{x\in X} |(f_i b v_i)(x)|^2\right)^{1/2} = \left(\sum_{i\in J}\left(\sum_{x\in X} |(f_i b v_i)(x)|^2\right)^{2/2}\right)^{1/2}$$

$$= \left(\sum_{i\in J} \|f_i b v_i\|_{\ell^2}^2\right)^{1/2} = \left(\sum_{j\in J}\left\|\left(\sum_{i\in J} f_i b v_i\right)_j\right\|_{\ell^2(X)}^2\right)^{1/2} = \left\|\sum_{j\in J} f_j b v_j\right\|_{\ell^2(J,\ell^2(X))}$$

as was to be shown. $\square$

Proposition A.2. Let $T, S \in \mathscr{B}(\mathcal{H})$ such that $\|T - S\| > \epsilon$. Then there exists unit vectors $\xi, \eta \in \mathcal{H}$ such that $|\langle (T-S)\xi, \eta\rangle| > \epsilon$.

Proof. Since $T - S \in \mathscr{B}(\mathcal{H})$,

$$\|T - S\| = \sup_{\|\zeta\|=1} \|(T - S)\zeta\| = M > \epsilon, \text{ for some } M \in \mathbb{R}.$$

Thus, there exists a $\xi \in (\mathcal{H})_1$ such that

$$\|(T - S)\xi\| + (M - \epsilon) > \sup_{\|\zeta\|=1} \|(T - S)\zeta\| = M,$$

so that $\|(T - S)\xi\| > \epsilon$. Next, $\|(T - S)\xi\| = L > \epsilon$ for some $L \in \mathbb{R}$. Thus, $|\langle (T - S)\xi, (T - S)\xi\rangle| = L^2$ so that

$$\left|\left\langle (T - S)\xi, \frac{(T - S)\xi}{\|(T - S)\xi\|}\right\rangle\right| = \frac{L^2}{\|(T - S)\xi\|} = L > \epsilon.$$

Taking $\eta = \frac{(T-S)\xi}{\|(T-S)\xi\|}$ we have that $|\langle (T - S)\xi, \eta\rangle| > \epsilon$ for the unit vectors $\xi, \eta \in \mathcal{H}$. $\square$

Proposition A.3. Suppose that: $(P_j)_{j\in X}$ is an increasing net of finite rank projections in $\ell^\infty(X)$ converging strongly to the identity, $(S_j)_{j\in X}$ is a net in $\mathscr{B}(\mathcal{H})$ converging weakly to S, and for the unit vectors $\xi, \eta \in \ell^2(X)$ we have that $|\langle (T - S)\xi, \eta\rangle| > \epsilon$ for some fixed ϵ. Then there exists a j_0 such that $\|(TP_j - S_j)\xi\| > \epsilon$ whenever $j \geq j_0$.

Proof. First note that, since P_j converges strongly to the identity, $P_j\xi \to \xi$ in norm. Thus,

$$\lim_{j\in X} |\langle (TP_j - S_j)\xi, \eta\rangle| = \left|\lim_{j\in X} \langle TP_j\xi, \eta\rangle - \lim_{j\in X} \langle S_j\xi, \eta\rangle\right| = |\langle (T - S)\xi, \eta\rangle| = M > \epsilon$$

for some $M \in \mathbb{R}$. Hence, there exists a j_0 such that for all $j \geq j_0$ we have that

$$|\langle (TP_j - S_j)\xi, \eta\rangle - \langle (T - S)\xi, \eta\rangle| < M - \epsilon$$

so that

$$M = |\langle (T - S)\xi, \eta\rangle| < |\langle (TP_j - S_j)\xi, \eta\rangle| + (M - \epsilon) \iff |\langle (TP_j - S_j)\xi, \eta\rangle| > \epsilon$$

whenever $j \geq j_0$. Lastly, since $\|\eta\| = 1$ and by the Cauchy Schwarz inequality we have that,

$$\|(TP_j - S_j)\xi\|^2 = \langle (TP_j - S_j)\xi, (TP_j - S_j)\xi \rangle \langle \eta, \eta \rangle \geq |\langle (TP_j - S_j)\xi, \eta \rangle|^2 > \epsilon^2$$

Thus, taking roots on both sides yields the desired result.

□

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